

MINIMALITY VIA SECOND VARIATION FOR MICROPHASE SEPARATION OF DIBLOCK COPOLYMER MELTS

VESA JULIN AND GIOVANNI PISANTE

ABSTRACT. We consider a non local isoperimetric problem arising as the sharp interface limit of the Ohta-Kawasaki free energy introduced to model microphase separation of diblock copolymers. We perform a second order variational analysis that allows us to provide a quantitative second order minimality condition. We show that critical configurations with positive second variation are indeed strict local minimizers of the nonlocal perimeter. Moreover we provide, via a suitable quantitative inequality of isoperimetric type, an estimate of the deviation from minimality for configurations close to the minimum in the L^1 topology .

1. INTRODUCTION

In this note we are interested in performing a second order analysis for a nonlocal isoperimetric problem arising as a variational limit of the Ohta-Kawasaki functional introduced for a density functional theory for microphase separation of A/B diblock copolymers.

Among the several mean field approximation theories proposed to model the phase separation in diblock copolymer melts, the one derived by Ohta and Kawasaki in [15] turns out to be one of the most promising from the mathematical point of view. Let $\Omega \subset \mathbb{R}^n$ be a domain representing the volume occupied by the polymeric material. The free energy can be written as a nonlocal functional of Cahn-Hilliard type as

$$(1.1) \quad \mathcal{E}_\varepsilon(u) = \varepsilon \int_{\Omega} |\nabla u|^2 dx + \frac{1}{\varepsilon} \int_{\Omega} (u^2 - 1)^2 dx + \gamma_0 \int_{\Omega} \int_{\Omega} G(x, y)(u(x) - m)(u(y) - m) dx dy,$$

where $u \in H^1(\Omega)$ represents the density distribution of the monomers forming the copolymers, G is the Green's function of the Laplace operator with Neumann boundary conditions and

$$m := \int_{\Omega} u dx$$

is the difference of the phases' volume fractions. The formulation (1.1) was first introduced in [14] and for a derivation of the Ohta-Kawasaki density function theory from the self-consistent mean field theory we refer the reader to [6] and the references therein.

As pointed out in [6], depending on the molecular structure of the polymers there are several regimes of phase mixture. Nevertheless the presence of an observable phase separation occurs in the so called intermediate or strong segregation regimes, wherein the domain size is much larger than the interfacial length. This suggests that most of the features of the model can be described, from a mathematical point of view, by looking at the sharp interface limit of \mathcal{E}_ε as the thickness ε of the diffuse interface tends to zero. Thus we are lead to study the minimizers of the following energy functional, that arises as the

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Γ -limit of \mathcal{E}_ε in the L^1 topology,

$$(1.2) \quad \mathcal{E}(u) = \frac{8}{3}|Du|(\Omega) + \gamma_0 \int_{\Omega} \int_{\Omega} G(x, y)(u(x) - m)(u(y) - m) dx dy$$

where $u : \Omega \rightarrow \{-1, 1\}$ is a function of bounded variation and $|Du|(\Omega)$ denotes its total variation in Ω . The functional (1.2) can be described in a more geometric way that turns out to be more suitable for our analysis. Indeed identifying the function u with the set $E = \{x \in \Omega \mid u(x) = 1\}$ and using the properties of $G(x, y)$, we can rewrite $\mathcal{E}(u)$ as

$$(1.3) \quad J(E) = P_{\Omega}(E) + \gamma \int_{\Omega} |\nabla v_E|^2 dx,$$

where $P_{\Omega}(E)$ stands for the perimeter of E in Ω and v_E is the solution of

$$(1.4) \quad \begin{cases} -\Delta v_E = u_E - m & \text{in } \Omega, \\ \frac{\partial v_E}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

with the positions

$$u_E = \chi_E - \chi_{\Omega \setminus E} \quad \text{and} \quad m = \int_{\Omega} u_E dx.$$

We are thus lead to study the variational problem

$$(1.5) \quad \min \left\{ J(E) \mid E \subset \Omega, \int_{\Omega} u_E dx = m \right\}.$$

We explicitly observe that the condition $\int_{\Omega} u_E dx = m$ is nothing but a volume constraint since $|E| = \frac{1}{2}(m + |\Omega|)$.

The problem (1.5) has been presented in [7] as a mathematical paradigm for the phenomenon of energy-driven pattern formation associated with competing short and long-range interactions. Recently an increasing interest has been devoted to the second order variational analysis of energy functionals which exhibit this competing behaviour. Indeed it has been successfully applied to prove stability and minimality criteria in several contexts (see for instance [1, 5]). Directly related to our problem is the work by Ren and Wei and by Choksi and Sternberg (cfr. [16, 17, 18, 19, 20, 7]). In a series of papers, as a first step toward the validation of the conjectured periodicity of global minimizers of (1.3), they calculate the second variation of (1.3) at a critical configuration and, among several other applications, they construct examples of periodical critical configurations and find conditions under which their second variation is positive definite.

A different approach is taken in [1], where the authors consider the problem of minimizing the functional (1.3) in the periodic case, i.e., when Ω is the n -dimensional flat torus. They prove the local minimality of critical configurations which second variation is positive definite. In this direction goes also our present work, whose aim is to prove that a similar minimality criterion holds true for the problem (1.5). Indeed we prove in Theorem 2.3 that if E is a sufficiently regular critical point of (1.3) such that the quadratic form $\partial^2 J(E)$ associated to the second variation of J at E is positively defined, then there exists a constant $c_0 > 0$ such that

$$J(F) \geq J(E) + c_0 |F \Delta E|^2$$

for any admissible set $F \subset \Omega$ sufficiently close to E in the L^1 topology. This in particular implies that E is a local minimizer of J and in addition provides a quantitative estimate of the deviation from minimality for sets near to E . We remark that our result holds also in the case $\gamma = 0$ and thus we cover the case of volume constrained isoperimetric problem in a regular domain Ω .

Some comments are in order on the differences between the periodic case (studied in [1]) and the Neumann boundary case. Indeed if on one side working with the Neumann setting dispenses us from several technicalities needed to deal with the translation invariance of the functional J , on the other side new delicate arguments need to be introduced to deal with problems which arise when E touches $\partial\Omega$. We remark that in [1] the Neumann boundary case was considered only under a rather restrictive assumption that the set E does not intersect the boundary of Ω .

Finally we outline the structure of the paper. In the next section we introduce the notation and present the framework in order to precisely state our main results. In Section 3 we discuss some regularity results for ω -minimizers of the area functional and in particular a stability result for the regularity (cfr. Theorem 3.5) which will play an important role in the proof of the main theorem. Section 4 is devoted to the lengthy calculations of the second variation formula for regular sets satisfying the orthogonality condition on $\partial\Omega$. The proof of the main result, Theorem 2.3, is divided in sections 5 and 6. The scheme follows a well established path (see for instance [1, 5]). First we use the general second variation formula from Proposition 4.1 to prove the local minimality among regular sets which are close to the critical one in the $W^{2,p}$ -topology. The final result is then proved by contradiction using a penalization argument and exploiting the regularity theory of ω -minimizers.

2. PRELIMINARIES AND STATEMENT OF THE RESULTS

In this section we set up the basic notation, recall some preliminary results and present the statements of the main results. Throughout the paper $\Omega \subset \mathbb{R}^n$ is assumed to be a bounded domain with $C^{3,\alpha}$ -boundary, for some $\alpha > 0$.

We say that a set $E \subset \Omega$ is $C^{k,\alpha}$ regular, where $k \leq 3$, if its relative boundary M

$$(2.1) \quad M := \overline{\partial E \cap \Omega}$$

is a $C^{k,\alpha}$ -manifold with or without boundary if $\partial E \cap \partial\Omega \neq \emptyset$ or $\partial E \cap \partial\Omega = \emptyset$ respectively. Suppose E is C^1 regular and let X be C^1 -vector field in Ω which satisfies

$$(2.2) \quad X(x) \in T_x(\partial\Omega) \quad \text{for every } x \in \partial\Omega,$$

and

$$(2.3) \quad \int_E \operatorname{div}(X) dx = \int_M \langle X, \nu_M \rangle d\mathcal{H}^{n-1} = 0,$$

where ν_M is unit normal of M . We define the associated flow $\Phi : \Omega \times (-t_0, t_0) \rightarrow \Omega$ by

$$\frac{\partial}{\partial t} \Phi(x, t) = X(\Phi(x, t)), \quad \Phi(x, 0) = x.$$

We then define the first variation of (1.3) at E with respect to the field X (or flow Φ) by

$$\frac{d}{dt} J(E_t) \Big|_{t=0}$$

and the second variation by

$$\frac{d^2}{dt^2} J(E_t) \Big|_{t=0},$$

where $E_t = \Phi(E, t)$.

Formula for the first variation of (1.3) is well known.

Proposition 2.1. *Let $E \subset \Omega$ be a C^1 -regular set with $\int_{\Omega} u_E dx = m$. Suppose that X satisfies (2.2) and (2.3). The first variation of (1.3) at E with respect to X can be written as*

$$\frac{dJ(E_t)}{dt}\Big|_{t=0} = \int_M (H_M + 4\gamma v_E) \langle X, \nu_M \rangle d\mathcal{H}^{n-1} + \int_{M \cap \partial\Omega} \langle X, \nu^* \rangle d\mathcal{H}^{n-2}.$$

Here ν^* is the outward unit co-normal of $M \cap \partial\Omega$, i.e., normal to $M \cap \partial\Omega$ and tangent to M , and ν_{Ω} is the unit normal of Ω .

We say that C^1 regular set $E \subset \Omega$ is *critical* if the first variation is zero for every C^1 -vector field X which satisfies (2.2) and (2.3).

If we choose the vector field X such that it is compactly inside Ω , $\text{spt}(X) \Subset \Omega$, then the last term in the first variation is trivially zero. Therefore if E is critical it satisfies the Euler-Lagrange equation in a weak sense

$$(2.4) \quad H_M + 4\gamma v_E = \lambda \quad \text{on } M,$$

where the constant λ is Lagrange multiplier associated to the volume constraint. We may further deduce that also the second term in the first variation vanishes

$$\int_{M \cap \partial\Omega} \langle X, \nu^* \rangle d\mathcal{H}^{n-2} = 0.$$

This and (2.2) imply that M is orthogonal to $\partial\Omega$ on $M \cap \partial\Omega$. These observations motivate us to define critical sets for (1.3).

Definition 2.2. We say that C^1 regular set $E \subset \Omega$ is a *regular critical set* if it satisfies (2.4) in a weak sense and meets $\partial\Omega$ orthogonally, i.e.,

$$(2.5) \quad \langle \nu_M, \nu_{\Omega} \rangle = 0 \quad \text{on } M \cap \partial\Omega.$$

We may use standard regularity theory for elliptic equations, as in [1], to deduce that every regular critical set is a $C^{1,\alpha}$ -manifold with boundary. Since v_E solves (1.4) we obtain that $v_E \in C^{1,\beta}(\Omega)$ for every $\beta \in (0, 1)$. Standard Schauder estimate then implies that M is in fact $C^{3,\beta}$ -manifold with boundary for every $\beta \in (0, 1)$.

It is well known that the second variation of (1.3) at a regular critical point has a special structure and it can be written as a quadratic form. To present the formula we remark that the non-local part of (1.3) can be written more explicitly in terms of E by using Green's function. Let $G : \Omega \times \Omega \rightarrow \mathbb{R}$ be the Green's function with Neumann boundary conditions

$$\begin{cases} -\Delta_y G(x, y) = \delta_x - \frac{1}{|\Omega|} & \text{in } \Omega, \\ \frac{\partial G(x, \cdot)}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} G(x, y) dy = 0. \end{cases}$$

The functional (1.3) can be written as

$$J(E) := P_{\Omega}(E) + \gamma \int_{\Omega} \int_{\Omega} G(x, y) u_E(x) u_E(y) dy dx.$$

Derivation of the second variation formula for critical sets can be found in [7] and [22], as it was noted after Remark 2.8 in [7]. To present the formula let $E \subset \Omega$ be a regular critical set and denote its

relative boundary by M . The second variation of (1.3) at E with respect to X can be written as

$$\begin{aligned}
 \partial^2 J(E)[\varphi] &= \int_M (|D_\tau \varphi|^2 - |B_M|^2 \varphi^2) d\mathcal{H}^{n-1} - \int_{M \cap \partial\Omega} B_{\partial\Omega}(\nu_M, \nu_M) \varphi^2 d\mathcal{H}^{n-2} \\
 (2.6) \quad &+ 8\gamma \int_M \int_M G(x, y) \varphi(x) \varphi(y) d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \\
 &+ 4\gamma \int_M \langle \nabla v_E, \nu_M \rangle \varphi^2 d\mathcal{H}^{n-1},
 \end{aligned}$$

where $\varphi = \langle X, \nu_M \rangle \in H^1(M)$ with $\int_M \varphi d\mathcal{H}^{n-1} = 0$ by (2.3). Here $B_{\partial\Omega}$ stands for the second fundamental form of $\partial\Omega$, i.e., the Hessian of the distance function from the boundary and $|B_M|^2$ is the sum of the square of the principal curvatures of M . If φ is not continuous the boundary term $\int_{M \cap \partial\Omega} B_{\partial\Omega}(\nu_M, \nu_M) \varphi^2 d\mathcal{H}^{n-2}$ is defined by the trace operator. We will return to the subject of second variation in section 4 and the role of the function φ .

It is rather straightforward to show that if C^1 regular set E is a local minimizer of (1.3), i.e., there is $\delta > 0$ such that $J(F) \geq J(E)$ for every $|F \Delta E| \leq \delta$, then the second variation is positive semi-definite

$$\partial^2 J(E)[\varphi] \geq 0$$

for every $\varphi \in H^1(\partial E)$ with $\int_{\partial E} \varphi d\mathcal{H}^{n-1} = 0$. The proof for this result is very similar to ([1], Corollary 3.4) and will be omitted here. As it was pointed out in the introduction, our main result Theorem 2.3, deals with the question whether we may conclude the stability from the formula (2.6).

Our goal is to prove that a regular critical point, i.e. a critical point with C^1 -boundary, with positive second variation is indeed a isolated local minimum.

Theorem 2.3. *Suppose that $E \subset \Omega$ with $\int_\Omega u_E dx = m$ is a regular critical point of (1.3) which satisfies*

$$\partial^2 J(E)[\varphi] > 0$$

for every $\varphi \in H^1(\partial E)$ with $\int_{\partial E} \varphi d\mathcal{H}^{n-1} = 0$. Then E is a strict local minimum and there are $c > 0$ and $\delta > 0$ such that

$$J(F) \geq J(E) + c_0 |F \Delta E|^2$$

for every $F \in BV(\Omega)$ with $\int_\Omega u_F dx = m$ and $|F \Delta E| \leq \delta$.

In order to prove Theorem 2.3 we need to calculate the second variation at any regular set which satisfies the orthogonality condition, not just for critical ones. This is done in Proposition 4.1 and thus we obtain the formula (2.6) as a corollary.

It is well known that the functionals \mathcal{E}_ε Γ -converges in L^1 to \mathcal{E} . As a corollary we obtain the following stability result.

Corollary 2.4. *Assume E is a regular critical point of J with positive second variation and denote $u_E = \chi_E - \chi_{\Omega \setminus E}$. There is ε_0 and a family $\{u_\varepsilon\}_{\varepsilon < \varepsilon_0}$ of isolated local minimizers of \mathcal{E}_ε with constraint $\int_\Omega u_\varepsilon dx = m$ such that $u_\varepsilon \rightarrow u_E$ in $L^1(\Omega)$ as $\varepsilon \rightarrow 0$.*

3. CRITICAL SETS AND REGULARITY OF ω -MINIMIZERS

Let us briefly discuss about the regularity of local minimizers of (1.3). Let us assume that we know, a priori, that our local minimizer is C^1 regular then we may use the Euler-Lagrange equation (2.4) to deduce that it is in fact $C^{3,\alpha}$ regular, as we pointed out in the previous section. Since our attention is in local minimizers, not in critical points, we will use the minimality itself to obtain the first order regularity. This will be done by using the theory of ω -minimizers (of area).

Definition 3.1. A set $E \in BV(\Omega)$ is an ω -minimizer with constants $\Lambda > 0$ and $r > 0$ if for every $G \subset \Omega$ with $G\Delta E \subset B_r(x_0)$ it holds

$$P_{B_r(x_0)}(E) \leq P_{B_r(x_0)}(G) + \Lambda |G\Delta E|.$$

We remark that the above definition differs slightly from the standard definition of ω -minimizer which appears in the literature e.g. in [11] and [23], where the measure of the symmetric difference $|G\Delta E|$ is replaced by the volume of the ball $|B_r|$. Definition 3.4 is clearly stronger and we may therefore apply known results from previously mentioned works. The motivation for the stronger definition is that for regular ω -minimizer we obtain a curvature bound.

Proposition 3.2. *Suppose that $E \in BV(\Omega)$ is an ω -minimizer with constants $\Lambda > 0$ and $r > 0$ and that E is C^1 regular. Then there exists a bounded function f such that E satisfies the equation*

$$(3.1) \quad H_M(x) = f(x) \quad x \in M \cap \Omega$$

in a weak sense. Moreover it holds $\|f\|_{L^\infty} \leq \Lambda$.

Proof. We may locally write M as a graph of a C^1 -function. Suppose that $\phi : D \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is such a function and let $\eta \in C_0^1(D)$. From the ω -minimizing property we obtain for every t it holds

$$\int_D \sqrt{1 + |\nabla \phi|^2} dz \leq \int_D \sqrt{1 + |\nabla \phi + t \nabla \eta|^2} dz + \Lambda t \int_D |\eta| dz.$$

Divide by t and let $t \rightarrow 0$ to obtain

$$\int_D \frac{\langle \nabla \phi, \nabla \eta \rangle}{\sqrt{1 + |\nabla \phi|^2}} dz \leq \Lambda \|\eta\|_{L^1(D)}.$$

Arguing exactly as in Proposition 7.41 in [3] we conclude that E satisfies (3.1) for some f with $\|f\|_{L^\infty} \leq \Lambda$. \square

We return to the C^1 -regularity of ω -minimizers. The following regularity result in a smooth domain Ω follows from a work by Grüter [11].

Theorem 3.3. *Suppose that E is an ω -minimizer in Ω . Then ∂E is C^1 regular outside a singular set Γ with Hausdorff dimension $\dim_{\mathcal{H}}(\Gamma) \leq n - 8$ and $M = \overline{\partial E} \cap \overline{\Omega}$ meets $\partial\Omega$ orthogonally on $\partial\Omega \setminus \Gamma$.*

In [11] Grüter states the result for minimizers of the partitioning problem, i.e. (1.3) with $\gamma = 0$. However he proves the result first by showing that the solutions of the partitioning problem are ω -minimizers. Then he uses a result from [12] to prove the C^1 -regularity by using the fact that mean curvature is L^p -integrable with $p > n$ which follows from the corresponding Euler-Lagrange equation. Hence may apply this result to our ω -minimizers which by Proposition 3.2 have bounded mean curvature. The interior regularity follows from a work by Tamanini [23].

As in [1] (Lemma 2.6) we have the Lipschitz continuity of the non-local part, i.e., for two measurable sets $E, F \subset \Omega$ we have that

$$(3.2) \quad \left| \int_{\Omega} |\nabla v_F|^2 dx - \int_{\Omega} |\nabla v_E|^2 dx \right| \leq C |F\Delta E|$$

for some dimensional constant C . Here v_E and v_F are defined as in (1.4). We may use an argument from [11] to prove that local minimizers of (1.3) are ω -minimizers in the sense of Definition 3.4. Although the proof is an exact copy from [11] we include it for the convenience of the reader.

Proposition 3.4. *Suppose that $E \in BV(\Omega)$ is a local minimizer of (1.3). Then it is an ω -minimizer.*

Proof. Let $G \subset \Omega$ be such that $G\Delta E \subset B_r(x_0)$ for some $r > 0$. We may assume that $|G| \leq |E|$ for in the case $|G| \geq |E|$ we argue similarly. Find a ball $B_r(y_0)$, disjoint with $B_r(x_0)$, such that

$$|B_r(y_0) \setminus E| = |E \cap B_r(x_0)| - |G \cap B_r(x_0)|.$$

By choosing $\tilde{G} = G \cup B_r(y_0)$ we have that $|\tilde{G}| = |E|$. The local minimality of E yields $J(E) \leq J(\tilde{G})$ and by (3.2) we deduce

$$(3.3) \quad P_{B_r(x_0)}(E) + P(E \cap B_r(y_0)) \leq P_{B_r(x_0)}(G) + P(B_r(y_0)) + C|\tilde{G}\Delta E|.$$

We may use the isoperimetric inequality to estimate

$$P(E \cap B_r(y_0)) \geq P(B_r(y_0)) - C|B_r(y_0) \setminus E|.$$

Notice that $|\tilde{G}\Delta E| \leq 2|G\Delta E|$ and $|B_r(y_0) \setminus E| \leq |G\Delta E|$. Hence, (3.3) implies

$$P_{B_r(x_0)}(E) \leq P_{B_r(x_0)}(G) + \Lambda|G\Delta E|.$$

□

We may use the regularity theory for ω -minimizers and the equation (2.4) to obtain that every local minimizer of (1.3) $C^{3,\alpha}$ -regular outside a critical set Γ with $\dim_{\mathcal{H}}(\Gamma) \leq n-8$. This motivates us to define regular critical sets as critical sets with no singularities.

It is well known that in higher dimension $n \geq 8$ the singular set even of a minimal surface might not be empty and therefore the regularity for ω -minimizers is optimal. However by the flatness theorem of De Giorgi (see [10]) we obtain the full regularity if we know that the blow-up limit at every point on the minimal surface is flat enough. Since every blow-up limit of an ω -minimizer converges to a minimal cone we obtain the following convergence result which concludes the section. The result can be obtained by following [2], [13] and [21] as explained in [24] with a few modifications. See also [23].

Theorem 3.5. *Suppose that E_k are ω -minimizers in Ω with uniform constants Λ and r . Suppose that E has $C^{1,\alpha}$ boundary in Ω with $\alpha \in (0, \frac{1}{2})$. If $E_k \rightarrow E$ in L^1 then*

$$\partial E_k \rightarrow \partial E \quad \text{in } C^{1,\alpha}.$$

In particular, ∂E_k is $C^{1,\alpha}$ -regular when k is large.

4. SECOND VARIATION FORMULA

In this section we calculate the second variational of the functional (1.3). Our main task is to write the formula in a form where the quadratic structure appears. As in [1] we have to generalize the formulas from [7] and [22] and calculate the second variation at any regular set E , not necessarily critical, and with respect to any given flow.

We assume that $E \subset \mathbb{R}^n$ is C^2 regular by which we mean that $M = \overline{\partial E \cap \Omega}$ is a C^2 -manifold with boundary. As in section 2 we consider a vector field $X \in C^2(\Omega; \mathbb{R}^n)$ which satisfies the tangent condition (2.2) and the associated flow $\Phi : \Omega \times (-t_0, t_0) \rightarrow \Omega$,

$$\frac{\partial}{\partial t}\Phi_t = X(\Phi_t), \quad \Phi(x, 0) = x.$$

We define the second variation at E with respect to the flow Φ as

$$\frac{d^2}{dt^2}J(E_t)|_{t=0},$$

where $E_t = \Phi(E, t)$.

A major technical challenge are the calculations for the local part of the energy $P_\Omega(\cdot)$. We follow a slightly different path than ([4], Proposition 3.9) where the proof contains calculations for the general second variation formula of the perimeter in \mathbb{R}^n . Instead of differentiating the first variation formula in Proposition 2.1 we will first calculate the second derivative of $P_\Omega(E_t)$ as it is done e.g. in [10] and then use the divergence theorem in order to find the "right" formula. This saves us from differentiating the boundary term which appears in Proposition 2.1. Along the calculations it will become clear that we need to assume the orthogonality condition (2.5) on E in order to find the quadratic structure in its second variation. As we pointed out in the previous section the orthogonality condition appears in the regularity theory for ω -minimizers. It can thus be viewed as a part of regularity assumption on E .

We recall the divergence theorem on M

$$\int_M \operatorname{div}_M X \, d\mathcal{H}^{n-1} = \int_M H \langle X, \nu_M \rangle \, d\mathcal{H}^{n-1} + \int_{M \cap \partial\Omega} \langle X, \nu^* \rangle \, d\mathcal{H}^{n-2},$$

where ν^* is the unit co-normal of $M \cap \partial\Omega$, see Proposition 2.1. Imposing the orthogonality condition on E implies $\nu^* = \nu_\Omega$ on $M \cap \partial\Omega$. We may thus write the integration by parts formula for $f \in C^1(\Omega)$ as

$$\int_M f \operatorname{div}_M X \, d\mathcal{H}^{n-1} = - \int_M \langle D_\tau f, X \rangle \, d\mathcal{H}^{n-1} - \int_M H f \langle X, \nu_M \rangle \, d\mathcal{H}^{n-1} + \int_{M \cap \partial\Omega} f \langle X, \nu_\Omega \rangle \, d\mathcal{H}^{n-2},$$

where D_τ is the tangential derivative on M , i.e., $D_\tau f = Df - \langle Df, \nu_M \rangle \nu_M$.

Proposition 4.1. *Let $E \subset \mathbb{R}^n$ be C^2 regular set which satisfies the orthogonality condition (2.5) and denote $M = \overline{\partial E} \cap \Omega$. Suppose that X satisfies the tangent condition (2.2) and denote $X_\nu = \langle X, \nu_M \rangle \nu_M$ and $X_\tau = X - X_\nu$. The second variation of (1.3) at E with respect to X can be written as*

$$\begin{aligned} \frac{d^2 J(E_t)}{dt^2} \Big|_{t=0} &= \int_M \left(|D_\tau \langle X, \nu_M \rangle|^2 - |B_M|^2 \langle X, \nu_M \rangle^2 \right) d\mathcal{H}^{n-1} - \int_{M \cap \partial\Omega} B_{\partial\Omega}(\nu, \nu) \langle X, \nu \rangle^2 d\mathcal{H}^{n-2} \\ &\quad + 8\gamma \int_M \int_M G(x, y) \langle X(x), \nu_M \rangle \langle X(y), \nu_M \rangle d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \\ &\quad + 4\gamma \int_M \langle \nabla v_E, \nu_M \rangle \langle X, \nu_M \rangle^2 d\mathcal{H}^{n-1} - \int_M (H + 4\gamma v_E) \operatorname{div}_\tau (X_\tau(\langle X, \nu_M \rangle)) d\mathcal{H}^{n-1} \\ &\quad + \int_M (H + 4\gamma v_E) \operatorname{div}(X) \langle X, \nu_M \rangle d\mathcal{H}^{n-1}. \end{aligned}$$

Here $B_{\partial\Omega}$ stands for the second fundamental form of $\partial\Omega$, i.e., the Hessian of the distance function from the boundary and $|B_M|^2$ is the sum of the square of the principal curvatures of M .

Before the proof we remark that the above formula consists of the quadratic form (2.6) by defining $\varphi := \langle X, \nu_M \rangle$ and two additional terms. It turns out that both of these extra terms vanish when E is critical. Indeed, since the minimizing problem (1.5) is volume-constrained it is natural to assume that the flow Φ preserves the volume

$$|E_t| = |E|$$

where $E_t = \Phi(E, t)$. From the volume constraint we obtain the first order condition $\frac{d}{dt}|E_t| \Big|_{t=0} = 0$ which yields (2.3)

$$\int_E \operatorname{div}(X) \, dx = \int_M \langle X, \nu_M \rangle \, d\mathcal{H}^{n-1} = 0.$$

The second order condition $\frac{d^2}{dt^2}|E_t| \Big|_{t=0} = 0$ yields

$$\int_E \operatorname{div}(\operatorname{div}(X)X) \, dx = \int_M \operatorname{div}(X) \langle X, \nu_M \rangle \, d\mathcal{H}^{n-1} = 0.$$

These calculations are standard and can be found e.g. in [7]. Therefore if E is a regular critical set, i.e., $H + 4\gamma v_E$ is constant and the flow is volume preserving we obtain the quadratic form (2.6) with $\varphi = \langle X, \nu_M \rangle$. Arguing exactly as in ([1], Corollary 3.4) we deduce that every regular local minimizer of J satisfies $\partial^2 J(E)[\varphi] \geq 0$ for every $\varphi \in C^2(M)$ with $\int_M \varphi d\mathcal{H}^{n-1} = 0$.

We give now the proof of the Proposition 4.1.

Proof of the Proposition 4.1. Denote the acceleration vector field by $Z = \frac{\partial^2}{\partial t^2} \Phi_t|_{t=0} = DX[X]$. We treat the perimeter and the nonlocal part of the energy separately and denote

$$E(t) = P_{\partial\Omega}(E_t), \quad F(t) = \gamma \int_{\Omega} |\nabla v_{E_t}|^2 dx.$$

For simplicity we write $\nu = \nu_M$.

The value for $F''(0)$ is calculated in [7] in the periodic setting. However, as it was pointed out in Remark 2.8 in [7], the Neumann boundary case does not produce any new terms to it and the formula for $F''(0)$ is the same as in the periodic case. Recalling the condition (2.2) we have from [7]

$$(4.1) \quad \begin{aligned} F''(0) &= 8\gamma \int_M \int_M G(x, y) \langle X(x), \nu \rangle \langle X(y), \nu \rangle d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \\ &\quad + 4\gamma \int_M \operatorname{div}(v_E X) \langle X, \nu \rangle d\mathcal{H}^{n-1}. \end{aligned}$$

The value of $E''(0)$ is well known in the following form, [10] (Theorem 10.4),

$$E''(0) = \int_M (\operatorname{div}_M Z + (\operatorname{div}_M X)^2 + |(D_\tau X)^T \nu_M|^2 - \operatorname{Tr}(D_\tau X)^2) d\mathcal{H}^{n-1},$$

where $\operatorname{div}_M X = \operatorname{div} X - \langle DX\nu, \nu \rangle$ and $(D_\tau X)_{i,j} = \langle D_\tau X_j, e_i \rangle = DX - (DX\nu) \otimes \nu$. By the divergence theorem we have

$$(4.2) \quad E''(0) = \int_M |(D_\tau X)^T \nu|^2 + (\operatorname{div}_M X)^2 - \operatorname{Tr}((D_\tau X)^2) + H \langle Z, \nu \rangle d\mathcal{H}^{n-1} + \int_{M \cap \partial\Omega} \langle Z, \nu_\Omega \rangle d\mathcal{H}^{n-2},$$

where we also used the orthogonality to conclude $\nu^* = \nu_\Omega$ on $M \cap \partial\Omega$. Notice that this is the second variation formula for the perimeter $P_\Omega(E_t)$ given by any vector field X .

We have to manipulate (4.2) in order to find the quadratic structure in it. We proceed by writing $X = X_\nu + X_\tau$ where

$$X_\nu = \langle X, \nu \rangle \nu.$$

The goal is to decompose every term in (4.2) by using $X = X_\nu + X_\tau$. We will first treat the term $\int_M |(D_\tau X)^T \nu|^2 d\mathcal{H}^{n-1}$ in (4.2). Since $D\nu \nu = 0$ we have $(D_\tau X_\nu)^T \nu = D_\tau \langle X, \nu \rangle$ and therefore

$$(4.3) \quad \begin{aligned} |(D_\tau X)^T \nu|^2 &= |(D_\tau X_\nu)^T \nu|^2 + 2((D_\tau X_\nu)^T \nu) \cdot ((D_\tau X_\tau)^T \nu) + |(D_\tau X_\tau)^T \nu|^2 \\ &= |D_\tau \langle X, \nu \rangle|^2 + 2 D_\tau \langle X, \nu \rangle \cdot ((D_\tau X_\tau)^T \nu) + |(D_\tau X_\tau)^T \nu|^2. \end{aligned}$$

Since $\operatorname{div}_M X_\nu = H \langle X, \nu \rangle$ the second term $\int_M (\operatorname{div}_M X)^2 d\mathcal{H}^{n-1}$ can be written as

$$(4.4) \quad \begin{aligned} (\operatorname{div}_M X)^2 &= (\operatorname{div}_M X_\nu + \operatorname{div}_M X_\tau)^2 \\ &= H^2 \langle X, \nu \rangle^2 + 2H \langle X, \nu \rangle \operatorname{div}_M X_\tau + (\operatorname{div}_M X_\tau)^2. \end{aligned}$$

For the third term we will use the equalities $(D_\tau X_\nu)^2 = \langle X, \nu \rangle^2 (D\nu)^2 + (D_\tau \langle X, \nu \rangle \cdot \nu) \nu \otimes D_\tau \langle X, \nu \rangle$ and $(D_\tau X_\tau)(D_\tau X_\nu) = \langle X, \nu \rangle D\nu D_\tau X_\tau$. Hence, we deduce

$$(4.5) \quad \begin{aligned} \operatorname{Tr}(D_\tau X)^2 &= \operatorname{Tr}(D_\tau X_\nu)^2 + 2 \operatorname{Tr}((D_\tau X_\tau)(D_\tau X_\nu)) + \operatorname{Tr}(D_\tau X_\tau)^2 \\ &= \langle X, \nu \rangle^2 |B_M|^2 + 2 \langle X, \nu \rangle \operatorname{Tr}(D\nu D_\tau X_\tau) + \operatorname{Tr}(D_\tau X_\tau)^2. \end{aligned}$$

We treat the fourth term $\int_M H \langle Z, \nu \rangle d\mathcal{H}^{n-1}$ by writing (here we denote $Z_\tau = D(X_\tau)(X_\tau)$)

$$\begin{aligned}
 \langle Z, \nu \rangle &= \langle DX X, \nu \rangle = \langle (DX(X_\nu) + D(X_\nu)(X_\tau) + D(X_\tau)(X_\tau)), \nu \rangle \\
 &= \langle X, \nu \rangle \langle DX \nu, \nu \rangle + D_\tau \langle X, \nu \rangle \cdot X_\tau + \langle Z_\tau, \nu \rangle \\
 (4.6) \quad &= \langle X, \nu \rangle \operatorname{div} X - \langle X, \nu \rangle \operatorname{div}_M X + D_\tau \langle X, \nu \rangle \cdot X_\tau + \langle Z_\tau, \nu \rangle \\
 &= \langle X, \nu \rangle \operatorname{div} X - \operatorname{div}_M (X_\tau \langle X, \nu \rangle) - H \langle X, \nu \rangle^2 + 2X_\tau \cdot D_\tau \langle X, \nu \rangle + \langle Z_\tau, \nu \rangle,
 \end{aligned}$$

where in the last equality we have used $\operatorname{div}_M X_\nu = H \langle X, \nu \rangle$.

For the last term $\int_{M \cap \partial \Omega} \langle Z, \nu_\Omega \rangle d\mathcal{H}^{n-2}$ we notice that the tangent condition (2.2) on X implies $\langle X, \nu_\Omega \rangle = 0$ on $M \cap \partial \Omega$. In particular, $D \langle X, \nu_\Omega \rangle \cdot X = 0$ on $M \cap \partial \Omega$. Hence the last term becomes

$$(4.7) \quad \int_{M \cap \partial \Omega} \langle Z, \nu_\Omega \rangle d\mathcal{H}^{n-2} = \int_{M \cap \partial \Omega} \langle DX X, \nu_\Omega \rangle d\mathcal{H}^{n-2} = - \int_{M \cap \partial \Omega} \langle D\nu_\Omega X, X \rangle d\mathcal{H}^{n-2}.$$

From now on we will use the notation $\langle D\nu_\Omega X, X \rangle = B_{\partial \Omega}(X, X)$.

We use (4.3), (4.4), (4.5), (4.6) and (4.7) to rewrite (4.2) as

$$\begin{aligned}
 E''(0) &= \int_M |D_\tau \langle X, \nu \rangle|^2 - |B_M|^2 \langle X, \nu \rangle^2 d\mathcal{H}^{n-1} - \int_{M \cap \partial \Omega} B_{\partial \Omega}(X, X) d\mathcal{H}^{n-2} \\
 &\quad + 2 \int_M D_\tau \langle X, \nu \rangle \cdot ((D_\tau X_\tau)^T \nu) d\mathcal{H}^{n-1} + 2 \int_M H (\langle X, \nu \rangle \operatorname{div}_M X_\tau) d\mathcal{H}^{n-1} \\
 (4.8) \quad &\quad - 2 \int_M \langle X, \nu \rangle \operatorname{Tr} (D\nu D_\tau X_\tau) d\mathcal{H}^{n-1} + 2 \int_M H (X_\tau \cdot D_\tau \langle X, \nu \rangle) d\mathcal{H}^{n-1} \\
 &\quad + \int_M H (2X_\tau \cdot D_\tau \langle X, \nu \rangle + \langle X, \nu \rangle \operatorname{div} X - \operatorname{div}_M (X_\tau \langle X, \nu \rangle)) d\mathcal{H}^{n-1} \\
 &\quad + \int_M (\operatorname{div}_M X_\tau)^2 + |(D_\tau X_\tau)^T \nu|^2 - \operatorname{Tr} (D_\tau X_\tau)^2 + H \langle Z_\tau, \nu \rangle d\mathcal{H}^{n-1},
 \end{aligned}$$

where $Z_\tau = D(X_\tau)(X_\tau)$.

We continue by treating the mixed terms in (4.8). We integrate by parts the third term in (4.8) and deduce

$$\begin{aligned}
 &2 \int_M D_\tau \langle X, \nu \rangle \cdot ((D_\tau X_\tau)^T \nu) d\mathcal{H}^{n-1} \\
 (4.9) \quad &= -2 \int_M \langle X, \nu \rangle \operatorname{div}_M ((D_\tau X_\tau)^T \nu) d\mathcal{H}^{n-1} + 2 \int_{M \cap \partial \Omega} \langle X, \nu \rangle \langle (D_\tau X_\tau)^T \nu, \nu_\Omega \rangle d\mathcal{H}^{n-2} \\
 &= -2 \int_M \langle X, \nu \rangle \operatorname{div}_M ((D_\tau X_\tau)^T \nu) d\mathcal{H}^{n-1} - 2 \int_{M \cap \partial \Omega} \langle X, \nu \rangle \langle D\nu X_\tau, \nu_\Omega \rangle d\mathcal{H}^{n-2},
 \end{aligned}$$

where we have used $0 = D_\tau \langle X_\tau, \nu \rangle \cdot \nu_\Omega = \langle D_\tau X_\tau \nu_\Omega, \nu \rangle + \langle D\nu X_\tau, \nu_\Omega \rangle$.

For the fourth term in (4.8) we observe that, by the orthogonality condition, X_τ vanishes on $M \cap \partial \Omega$ and therefore integration by parts yields

$$\begin{aligned}
 (4.10) \quad &2 \int_M H \langle X, \nu \rangle \operatorname{div}_M X_\tau d\mathcal{H}^{n-1} = -2 \int_M D_\tau (H \langle X, \nu \rangle) \cdot X_\tau d\mathcal{H}^{n-1} + 2 \int_{M \cap \partial \Omega} H \langle X, \nu \rangle \langle X_\tau, \nu_\Omega \rangle d\mathcal{H}^{n-2} \\
 &= -2 \int_M H X_\tau \cdot D_\tau \langle X, \nu \rangle d\mathcal{H}^{n-1} - 2 \int_M \langle X, \nu \rangle D_\tau H \cdot X_\tau d\mathcal{H}^{n-1}.
 \end{aligned}$$

Next we introduce a notation δ_i for partial derivative on M , i.e., $\delta_i g = \partial_{x_i} g - \langle Dg, \nu \rangle \nu_i$ for any smooth function g . We use the facts $\delta_j \nu_i = \delta_i \nu_j$ and $\sum_{i,j}^n (\delta_j \delta_i \nu_i) X_\tau^j = \sum_{i,j}^n (\delta_i \delta_i \nu_j) X_\tau^j$ (Lemma 10.7 in

[10]) to obtain

$$\begin{aligned}
(4.11) \quad & \operatorname{div}_M ((D_\tau X_\tau)^T \nu) + D_\tau H \cdot X_\tau + \operatorname{Tr} (D\nu D_\tau X_\tau) \\
&= \sum_{i,j}^n \delta_i (\delta_i X_\tau^j \nu_j) + (\delta_j \delta_i \nu_i) X_\tau^j + \delta_j \nu_i \delta_i X_\tau^j \\
&= \sum_{i,j}^n \delta_i (\delta_i X_\tau^j \nu_j) + (\delta_i \delta_i \nu_j) X_\tau^j + \delta_i \nu_j \delta_i X_\tau^j \\
&= \sum_{i,j}^n \delta_i \delta_i (X_\tau^j \nu_j) = \Delta_M \langle X_\tau, \nu \rangle = 0,
\end{aligned}$$

where $\Delta_M = \sum_{i=1}^n \delta_i \delta_i$ is the Laplacian on M .

Together with (4.9), (4.10) and (4.11) the formula (4.8) becomes

$$\begin{aligned}
(4.12) \quad E''(0) &= \int_M |D_\tau \langle X, \nu \rangle|^2 - |B_M|^2 \langle X, \nu \rangle^2 d\mathcal{H}^{n-1} \\
&\quad - \int_{M \cap \partial\Omega} B_{\partial\Omega}(X, X) d\mathcal{H}^{n-2} - 2 \int_{M \cap \partial\Omega} \langle X, \nu \rangle \langle D\nu X_\tau, \nu_\Omega \rangle d\mathcal{H}^{n-2} \\
&\quad + \int_M H (\langle X, \nu \rangle \operatorname{div} X - \operatorname{div}_M (X_\tau \langle X, \nu \rangle)) d\mathcal{H}^{n-1} \\
&\quad + \int_M (\operatorname{div}_M X_\tau)^2 + |(D_\tau X_\tau)^T \nu|^2 - \operatorname{Tr}(D_\tau X_\tau)^2 + H \langle Z_\tau, \nu \rangle d\mathcal{H}^{n-1}.
\end{aligned}$$

Let us treat the last row in the previous formula. We claim that

$$(4.13) \quad \int_M (\operatorname{div}_M X_\tau)^2 + |(D_\tau X_\tau)^T \nu|^2 - \operatorname{Tr}(D_\tau X_\tau)^2 + H \langle Z_\tau, \nu \rangle d\mathcal{H}^{n-1} + \int_{M \cap \partial\Omega} \langle Z_\tau, \nu_\Omega \rangle d\mathcal{H}^{n-2} = 0.$$

Indeed, the formula (4.13) is nothing but the second variation formula (formula (4.2)) of the perimeter with respect to the vector field X_τ . As we noticed before X_τ vanishes on $M \cap \partial\Omega$ and therefore the flow associated by X_τ leaves M unchanged. Hence, we have (4.13).

Since X_τ vanishes on $M \cap \partial\Omega$, we have $D \langle X_\tau, \nu_\Omega \rangle \cdot X_\tau = 0$ on. This yields

$$(4.14) \quad \int_{M \cap \partial\Omega} \langle Z_\tau, \nu_\Omega \rangle d\mathcal{H}^{n-2} = \int_{M \cap \partial\Omega} \langle D(X_\tau) X_\tau, \nu_\Omega \rangle d\mathcal{H}^{n-2} = - \int_{M \cap \partial\Omega} B_{\partial\Omega}(X_\tau, X_\tau) d\mathcal{H}^{n-2}.$$

Moreover, since $\langle \nu, \nu_\Omega \rangle = 0$ on $M \cap \partial\Omega$ we get $0 = D \langle \nu, \nu_\Omega \rangle \cdot X_\tau = \langle D\nu X_\tau, \nu_\Omega \rangle + \langle D\nu_\Omega X_\tau, \nu \rangle$ and therefore

$$(4.15) \quad 2 \int_{M \cap \partial\Omega} \langle X, \nu \rangle \langle D\nu X_\tau, \nu_\Omega \rangle d\mathcal{H}^{n-2} = -2 \int_{M \cap \partial\Omega} \langle D\nu_\Omega X_\tau, X_\nu \rangle d\mathcal{H}^{n-2} = -2 \int_{M \cap \partial\Omega} B_{\partial\Omega}(X_\tau, X_\nu) d\mathcal{H}^{n-2}.$$

We combine (4.1) and (4.12), together with (4.13), (4.14) and (4.15) and write the second variation as

$$\begin{aligned}
(4.16) \quad & \frac{d^2 J(E_t)}{dt^2} \Big|_{t=0} = \int_M \left(|D_\tau \langle X, \nu_M \rangle|^2 - |B_M|^2 \langle X, \nu_M \rangle^2 \right) d\mathcal{H}^{n-1} - \int_{M \cap \partial\Omega} B_{\partial\Omega}(\nu, \nu) \langle X, \nu \rangle^2 d\mathcal{H}^{n-2} \\
& \quad + \int_M H (\langle X, \nu \rangle \operatorname{div} X - \operatorname{div}_M (X_\tau \langle X, \nu \rangle)) d\mathcal{H}^{n-1} \\
& \quad + 8\gamma \int_M \int_M G(x, y) \langle X(x), \nu \rangle \langle X(y), \nu \rangle d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \\
& \quad + 4\gamma \int_M \operatorname{div}(v_E X) \langle X, \nu \rangle d\mathcal{H}^{n-1}.
\end{aligned}$$

As in [1] we integrate by parts to obtain

$$\begin{aligned}
\int_M \operatorname{div}(v_E X) \langle X, \nu \rangle d\mathcal{H}^{n-1} &= \int_M \langle Dv_E, X \rangle \langle X, \nu \rangle d\mathcal{H}^{n-1} + \int_M v_E \operatorname{div}(X) \langle X, \nu \rangle d\mathcal{H}^{n-1} \\
&= \int_M \langle D_\tau v_E, X_\tau \rangle \langle X, \nu \rangle d\mathcal{H}^{n-1} + \int_M \langle \nabla v_E, \nu \rangle \langle X, \nu \rangle^2 d\mathcal{H}^{n-1} + \int_M v_E \operatorname{div}(X) \langle X, \nu \rangle d\mathcal{H}^{n-1} \\
&= - \int_M v_E \operatorname{div}(X_\tau \langle X, \nu \rangle) d\mathcal{H}^{n-1} + \int_M \langle \nabla v_E, \nu \rangle \langle X, \nu \rangle^2 d\mathcal{H}^{n-1} + \int_M v_E \operatorname{div}(X) \langle X, \nu \rangle d\mathcal{H}^{n-1},
\end{aligned}$$

where the last equality follows from the fact that X_τ vanishes on $M \cap \partial\Omega$. Combining this to (4.16) yields the result. \square

5. $W^{2,p}$ -MINIMALITY

In this section we prove that a regular critical set E with a positive second variation is a strict local minimizer among sets which are regular and close to E in a strong sense (in $W^{2,p}$ topology). This result is stated in Proposition 5.2 and is interesting in itself. The idea is to take an arbitrary set F near E and to construct a volume preserving flow Φ such that $\Phi(E, 1) = F$. We use Proposition 4.1 to estimate the second variation at $E_t = \Phi(E, t)$ at any time $t \in [0, 1]$. Notice that by the assumption on E we know that the second derivative of $t \mapsto J(E_t)$ at $t = 0$ is strictly positive. We then use the fact that E_t are close to E in $W^{2,p}$ topology to deduce that the function $t \mapsto J(E_t)$ is in fact strictly convex. We note that in order to use Proposition 4.1 all the sets have to satisfy the orthogonality condition.

We begin by a simple compactness argument. The proof is exactly the same as Lemma 3.6 in [1] and will therefore be omitted.

Lemma 5.1. *Suppose that E is a critical point with positive second variation. There is $c_0 > 0$ such that*

$$\partial^2 J(E)[\varphi] \geq c_0 \|\varphi\|_{H^1(\partial E)}^2$$

for every $\varphi \in H^1(\partial E)$ with $\int_{\partial E} \varphi d\mathcal{H}^{n-1} = 0$.

We define $W^{2,p}$ and C^2 distances between regular sets $E, F \subset \Omega$ as

$$\begin{aligned}
\|E, F\|_{W^{2,p}} &:= \inf\{\|\Phi - Id\|_{W^{2,p}(\Omega)} \mid \Phi : E \rightarrow F, \text{ } C^2\text{-diffeomorphism}\} \\
\|E, F\|_{C^2} &:= \inf\{\|\Phi - Id\|_{C^2(\Omega)} \mid \Phi : E \rightarrow F, \text{ } C^2\text{-diffeomorphism}\}.
\end{aligned}
\tag{5.1}$$

Here is the main result of the section.

Proposition 5.2. *Let E be as in Theorem 2.3 and $p > n$. There is $\delta > 0$ and a constant $c_1 > 0$ such that for every $F \subset \Omega$ with $|F| = |E|$ which satisfies the orthogonality condition and $\|E, F\|_{W^{2,p}} \leq \delta$ it holds*

$$J(F) \geq J(E) + c_1 |F \Delta E|^2.$$

We first prove some technical lemmata and then give the proof of Proposition 5.2 at the end of the section. Most important of the lemmata is Lemma 5.3 where we construct a vector field X and a flow Φ which takes the given set E to a regular set F nearby. The proof is technical and long and it is therefore divided into three steps. The difficulty lies in the fact that the flow needs to satisfy both orthogonality and the volume constraint. We divide the proof by first constructing a primitive flow which satisfies the orthogonality condition and then, at the third step, we correct it so that finally also the volume constraint is satisfied.

Lemma 5.3. *Let E be as in Theorem 2.3. Suppose F is C^2 regular set which satisfies the volume constraint $|F| = |E|$, the orthogonality condition (2.5) and $\|E, F\|_{W^{2,p}} \leq \delta$, where $p > n$. When δ is small enough there is C^2 regular vector field X with uniformly bounded $W^{2,p}$ -norm and associated flow Φ*

$$\frac{\partial}{\partial t}\Phi(x, t) = X(\Phi(x, t)), \quad \Phi(x, 0) = x,$$

such that at every $t \in [0, 1]$ the set $E_t = \Phi(E, t)$ satisfies the volume constraint $|E_t| = |E|$, and the orthogonality condition (2.5). Moreover the flow takes E to F , i.e., $E_1 = F$.

Proof. Step 1: First we construct a primitive vector field Z which has the property that the associated flow $\Phi_Z : \Omega \times (-t_0, t_0) \rightarrow \Omega$,

$$\frac{\partial}{\partial t}\Phi_Z = Z(\Phi_Z) \quad \text{with } \Phi_Z(x, 0) = x$$

is such that, at any $t \in (-t_0, t_0)$, the set $\Phi_Z(E, t)$ satisfies the orthogonality condition. We will build the vector field only in a neighbourhood of the relative boundary $M = \overline{\partial E \cap \Omega}$, which we denote by U . By neighbourhood of M we mean a connected set U which contains M and is open with respect to $\bar{\Omega}$. We begin by defining a function which plays the role of a signed distance function of the set E with respect to the domain Ω .

Since E is $C^{3,\alpha}$ regular and satisfies the orthogonality condition there is a $C^{3,\alpha}$ regular function $d_{\Omega,E} : U \rightarrow \mathbb{R}$ which satisfies

- (i) $d_{\Omega,E} = 0$ on M and $d_{\Omega,E} \leq 0$ in E ,
- (ii) $|\nabla d_{\Omega,E}| \geq c > 0$ in U ,
- (iii) $\frac{\partial d_{\Omega,E}}{\partial \nu_\Omega} = 0$ on $\partial\Omega \cap U$.

Such a function can be found e.g. by minimizing the energy

$$\int_{U \setminus E} |Du|^2 dx \quad u \in W^{1,2}(U \setminus E)$$

with respect to Dirichlet boundary condition $u = 0$ on M and $u = 1$ on $(\partial U \cap \Omega) \setminus E$. Here we assume without loss of generality that the boundary $\partial U \cap \Omega$ meets $\partial\Omega$ orthogonally. We may choose $d_{\Omega,E}$ in $U \setminus E$ to be the minimizer of the above problem. One may then extend this function naturally to whole U .

It follows from the above conditions (i) – (iii) that for small values of $t \in \mathbb{R}$ the level sets

$$\{x \in \Omega \mid d_{\Omega,E}(x) = t\}$$

form a foliation of U with $C^{3,\alpha}$ regular sets which meet $\partial\Omega$ orthogonally by the Neumann boundary condition (iii). We will define the primitive flow Φ_Z such that

$$\Phi_Z(E, t) = \{x \in \Omega \mid d_{\Omega,E}(x) \leq t\}.$$

For future purposes we need yet to construct the trajectories of the flow.

To that aim we construct a vector field Z' which defines the trajectories. We begin by defining

$$Z_1 := \nabla d_{\Omega,E} \quad \text{in } U.$$

To make sure that we will not loose the orthogonality in the forthcoming steps we correct Z_1 near the boundary $\partial\Omega$. This will be done by "copying" the geometry of $\partial\Omega$ into its neighbourhood. We denote the neighbourhood of $\partial\Omega$ by U' .

Since $\partial\Omega$ is regular we may define the standard projection to $\partial\Omega$, $\Pi : U' \rightarrow \partial\Omega$, as $\Pi(x) = y_x$, where $y_x \in \partial\Omega$ is the unique point for which

$$d(x, \Omega) = \inf_{y \in \Omega} |x - y| = |x - y_x|.$$

Every point $x \in U'$ can therefore be written as $x = \Pi(x) - d(x, \Omega)\nu_\Omega(\Pi(x))$ where $\nu_\Omega(y)$ is the outer normal of Ω at $y \in \partial\Omega$. We define the field Z_2 such that $Z_2 = Z_1$ on the boundary $\partial\Omega \cap U$ and

$$Z_2(y) = Z_1(\Pi(y)) \quad \text{for } y \in U \cap U'.$$

The advantage of Z_2 is the following geometrical fact. If we know that the points $x, y \in \partial\Omega \cap U$ lie on the same trajectory, then for small $h > 0$ also the points $x - h\nu_\Omega(x)$ and $y - h\nu_\Omega(y)$ lie on the same trajectory. We will use this fact frequently.

Finally we define Z' as

$$Z' = \eta Z_1 + (1 - \eta)Z_2$$

where $\eta \in C_0^\infty(\Omega)$ is a standard cut-off function such that $\eta \equiv 0$ near $\partial\Omega$ and $\eta \equiv 1$ outside U' . By choosing the cut-off function properly we can make sure that Z' is close to $\nabla d_{\Omega, E}$, i.e.,

$$\|Z' - \nabla d_{\Omega, E}\|_{L^\infty(M)} < \varepsilon.$$

For every point $x \in M$ we define the primitive flow $x(t)$ such that the trajectories are given by Z' and $x(t)$ is the unique point on the intersection of the level set $\{x \in \Omega \mid d_{\Omega, E}(x) = t\}$ and the trajectory passing through x . In other words $x(\cdot)$ is such that $x(0) = x$ and

$$(5.2) \quad d_{\Omega, E}(x(t)) = t.$$

We denote the flow by $\Phi_Z(x, t)$ and the associated vector field by Z . Since Z' is close to $\nabla d_{\Omega, E}$ on $M = \{x \mid d_{\Omega, E}(x) = 0\}$ we have that

$$(5.3) \quad |\langle Z(x), \nu_M \rangle| \geq (1 - \varepsilon)|Z(x)| \quad \text{for all } x \in M.$$

The flow Φ_Z satisfies the orthogonality condition by construction. In fact the construction implies that every point $y \in \partial\Omega \cap U$ can be written as $y = \Phi_Z(x, t)$ for some $t \in (-t_0, t_0)$ and it holds

$$(5.4) \quad D\Phi_Z(x, t)\nu_\Omega(x) = \nu_\Omega(y).$$

This is a priori a stronger than the orthogonality condition since it implies that the vector $D\Phi_Z(x, t)\nu_\Omega(x)$ is the unit co-outer normal of the set $\Phi_Z(M, t)$ and it equals $\nu_\Omega(\Phi_Z(x, t))$.

To verify (5.4) write $y = \Phi_Z(x, t)$ and notice that by the construction, for every small $h > 0$, the points $x - h\nu_\Omega(x)$ and $y - h\nu_\Omega(y)$ lie on the same trajectory. Therefore there is t_h such that

$$y - h\nu_\Omega(y) = \Phi_Z(x - h\nu_\Omega(x), t_h).$$

In particular, we deduce from (5.2) that $d_{\Omega, E}(y - h\nu_\Omega(y)) = t_h$.

Notice that $d_{\Omega, E}(y) = d_{\Omega, E}(\Phi_Z(x, t)) = t$. Therefore by the orthogonality of the set $\{x \mid d_{\Omega, E}(x) = t\}$ it holds

$$d_{\Omega, E}(y - h\nu_\Omega(y)) = t + o(h).$$

Consequently $t_h = t + o(h)$ and (5.4) follows.

Step 2: Next we change Z to Y so that the new flow, denoted by Φ_Y , satisfies $\Phi_Y(E, 1) = F$ and the orthogonality is preserved.

In the neighbourhood U of M the map $\Phi_Z|_{M \times (-t_0, t_0)} \rightarrow U$ is $C^{3,\alpha}$ -diffeomorphism and every point $y \in U$ can be written as $y = \Phi_Z(x, t)$ for some $x \in M$ and for $t = d_{\Omega, E}(y)$. We may therefore define implicitly a projection $\pi : U \rightarrow M$ such that

$$\Phi_Z(\pi(y), d_{\Omega, E}(y)) = y.$$

Since Z and $d_{\Omega, E}$ are $C^{3,\alpha}$ regular, also π is $C^{3,\alpha}$ regular.

By the assumption on F there is C^2 -diffeomorphism $\Psi : E \rightarrow F$ with $\|\Psi - Id\|_{W^{2,p}} < \delta$. Without further mention we always assume $\delta > 0$ to be small. We define a map $S : M \rightarrow M$ as

$$S(x) = \pi(\Psi(x)).$$

The tangential differential of S is

$$D_\tau S(x) = D_\tau \pi(\Psi(x)) D_\tau \Psi(x) \quad \text{on } x \in M.$$

From the regularity of π and from the fact $\pi(x) = x$ for $x \in M$ we conclude that $|D_\tau \pi(\Psi(x))| \geq c$. Since $\|D_\tau \Psi(x) - I\|_{L^\infty} \leq \delta$ we conclude that S is C^2 -diffeomorphism and the bound $\|\Psi\|_{W^{2,p}} \leq C$ implies

$$(5.5) \quad \|S^{-1}\|_{W^{2,p}(M)} \leq C.$$

We define further a projection on F $M_F := \overline{\partial F \cap \Omega}$, $\pi_F : U \rightarrow M_F$, as

$$\pi_F(y) = \Psi(S^{-1}(\pi(y)))$$

which labels for every point $y \in U$ a unique point z_y on M_F on the same trajectory. Trivially the map π_F is constant along the trajectories of Φ_Z and the value $T_F(y) := d_{\Omega, E}(\pi_F(y))$ denotes the time needed from M to M_F along the trajectory passing through y . We define the vector field Y by

$$(5.6) \quad Y(y) = T_F(y)Z(y)$$

and denote the associated flow by Φ_Y . We note that we may write Φ_Y explicitly as

$$(5.7) \quad \Phi_Y(x, t) = \Phi_Z(x, T_F(x)t) \quad \text{for } x \in M.$$

We denote by F_t the set enclosed by $\Phi_Y(M, t)$. Since $T_F(x)$ is the time needed from $x \in M$ to M_F it follows that $M_F = \{\Phi_Y(x, 1) \mid x \in M\}$ and therefore $F_1 = F$. Moreover we have that $\|T_F\|_{W^{2,p}} \leq \|F, E\|_{W^{2,p}} \leq \delta$.

We have to make sure that at any time $t \in (0, 1)$ the set F_t satisfies the orthogonality condition. As in the previous step we will show that for any $y \in \Phi_Y(M, t) \cap \partial\Omega$ it holds

$$(5.8) \quad D\Phi_Y(x, t)\nu_\Omega(x) = \nu_\Omega(y)$$

where $y = \Phi_Y(x, t)$ for some $x \in M$. This implies the orthogonality.

To show (5.8) we use (5.7) to calculate

$$D\Phi_Y(x, t)\nu_\Omega(x) = D\Phi_Z(x, T_F(x)t)\nu_\Omega(x) + t \frac{\partial \Phi_Z}{\partial t}(x, T_F(x)(x)t) \langle \nabla T_F(x), \nu_\Omega(x) \rangle.$$

Hence by (5.4) we need to show

$$(5.9) \quad \langle \nabla T_F(x), \nu_\Omega(x) \rangle = 0 \quad \text{for } x \in M \cap \partial\Omega.$$

Since $T_F(y) = d_{\Omega, E}(\pi_F(y))$, (5.9) can be written as

$$\langle \nabla d_{\Omega, E}(\pi_F(x)), D\pi_F(x)\nu_\Omega(x) \rangle = 0 \quad \text{for } x \in M \cap \partial\Omega.$$

We claim that $D\pi_F(x)\nu_\Omega(x) = \nu_\Omega(\pi_F(x))$. Then (5.9) follows from the fact that $d_{\Omega,E}$ satisfies the Neumann boundary condition

$$\frac{\partial d_{\Omega,E}}{\partial \nu_\Omega} = 0.$$

To show $D\pi_F(x)\nu_\Omega(x) = \nu_\Omega(y)$ where $y = \pi_F(x)$, recall that π_F is a map which labels for every $x' \in U$ a unique point $y' \in M_F$ on the same trajectory. Therefore the points $x - h\nu_\Omega(x)$, $y - h\nu_\Omega(y)$ and $\pi_F(x - h\nu_\Omega(x))$ all lie on the same trajectory. It follows from the orthogonality of F that

$$\pi_F(x - h\nu_\Omega(x)) = y - h\nu_\Omega(y) + o(h)$$

and the claim follows.

At the end of the step we will write down some useful properties of the flow Φ_Y and the field Y for the next step. First of all, it follows from the definition of Y (5.6), (5.3), (5.5) and the fact that Z is C^2 regular that

$$(5.10) \quad C^{-1}|T_F| \leq |Y| \leq C|T_F| \quad \text{and} \quad \|Y\|_{W^{2,p}} \leq C\|T_F\|_{W^{2,p}} \leq \delta.$$

In particular since T_F is constant along the trajectories (5.10) yields

$$(5.11) \quad C^{-1}|T_F(x)| \leq |Y(\Phi_Y(x, t))| \leq C|T_F(x)| \quad \text{for all } x \in M$$

and we may also estimate the divergence by

$$(5.12) \quad |\operatorname{div}(Y)| = |T_F \operatorname{div}(Z)| \leq C|Y|.$$

Since Φ_Y solves

$$\frac{\partial}{\partial t} \Phi_Y(x, t) = Y(\Phi_Y(x, t)), \quad \text{and} \quad \Phi_Y(x, 0) = x$$

we may estimate, by differentiating the above equation for Φ_Y twice and by using the second inequality in (5.10), that

$$\|\Phi_Y(\cdot, t) - Id\|_{W^{2,p}(M)} \leq C\|T_F\|_{W^{2,p}} \leq \delta.$$

The above estimate together with (5.3) imply

$$(5.13) \quad |\langle Y, \nu_{\partial F_t} \rangle| \geq (1 - 2\varepsilon)|Y|.$$

Step 3: We modify Y to X so that the new flow will finally be volume preserving. We note that the sets

$$F_t = \Phi_Y(E, t)$$

satisfy everything except the volume constraint but the final set $|F_1| = |F|$ has the right volume. The idea is to study the evolution of $F_t \setminus E$ and $E \setminus F_t$. If the volumes of these two sets do not match we give the other one a little boost. More precisely we would like to find for every $t \in [0, 1]$ a value $s(t)$ such that

$$|F_{t+s(t)} \setminus E| = |E \setminus F_t|.$$

The problem is that this would break the regularity of the flow.

To overcome this problem we choose a smooth function f defined on M such that $f \simeq \chi_{M \cap F}$. To be more precise, we choose f such that it satisfies

- (i) $f = 1$ on $M \cap F$ and $f < 1$ on $M \setminus F$,
- (ii) $f = 0$ on $M \setminus F_\varepsilon$ where $F_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, F) < \varepsilon\}$,
- (iii) $\langle \nabla f, \nu_\Omega \rangle = 0$ on $M \cap \partial\Omega$.

For every $t \in [0, 1]$ we define a flow $\tilde{\Phi}_t : M \times [-t, 1-t] \rightarrow \Omega$ such that

$$\tilde{\Phi}_t(x, s) = \Phi_Y(x, t + sf(x))$$

and the set enclosed by $\tilde{\Phi}_t(M, s)$ by $F_{t,s}$. We denote the associated vector field by \tilde{X}_t which we may write as

$$\tilde{X}_t(y) = f(\pi(y)) Y(y)$$

where π is the projection on M defined in the beginning of step 2. We note that $F_{t,0} = F_t$ is the set given by the flow Φ_Y . Moreover since $f \equiv 1$ on $M \cap F$ we have

$$(5.14) \quad \tilde{\Phi}_t(x, s) = \Phi_Y(x, t + s) \quad \text{for } x \in M \cap F.$$

The idea is to correct the volume of F_t such that the new set, denoted by E_t , will satisfy the volume constraint. To that aim we define a function

$$\varphi(t, s) := |F_{t,s}| - |E|.$$

The correction value $s(t)$ is then implicitly defined as $\varphi(t, s(t)) = 0$ and $E_t := F_{t,s(t)}$. To be more rigorous for every $t \in [0, 1]$ we define $s(t)$ as

$$(5.15) \quad s(t) := \sup\{s \in [-t, 1-t] : \varphi(t, s) \leq 0\}.$$

First we have to make sure that $s(t)$ is well defined.

Firstly we show that for every $t \in (0, 1)$ the function $s \mapsto \varphi(t, s)$ is strictly increasing, i.e.,

$$(5.16) \quad \frac{\partial \varphi(t, s)}{\partial s} > 0.$$

From (5.11) and (5.13) we conclude that the growth of the volume $|F_t \setminus E|$ do not degenerate

$$(5.17) \quad \frac{d}{dt}|F_t \setminus E| = \int_{\partial F_t \setminus E} \langle Y, \nu_{F_t} \rangle d\mathcal{H}^{n-1} \simeq \int_{M \cap F} |T_F(x)| d\mathcal{H}^{n-1}.$$

By notation $\alpha \simeq \beta$ we mean that $C^{-1}\alpha \leq \beta \leq C\alpha$ for some constant C . In particular the above estimate implies

$$(5.18) \quad |F \Delta E| = 2|F \setminus E| = 2 \int_0^1 \frac{d}{dt}|F_t \setminus E| dt \simeq \int_{M \cap F} |T_F(x)| d\mathcal{H}^{n-1}.$$

Similarly

$$(5.19) \quad \frac{d}{dt}|E \setminus F_t| = \int_{\partial F_t \cap E} |\langle Y, \nu_{F_t} \rangle| d\mathcal{H}^{n-1} \simeq \int_{M \setminus F} |T_F(x)| d\mathcal{H}^{n-1} \simeq |F \Delta E|.$$

Since the vector field \tilde{X}_t has the same regularity as Y we may estimate the growth of $|E \setminus F_{t,s}|$ for every $t \in (0, 1)$ and every $s \in (-t, 1-t)$ exactly as we with (5.17) and obtain

$$\frac{\partial}{\partial s}|E \setminus F_{t,s}| = - \int_{\partial F_{t,s} \cap E} \langle \tilde{X}_t, \nu_{F_{t,s}} \rangle d\mathcal{H}^{n-1} \leq C \int_{M \setminus F} f(x)|Y(x)| d\mathcal{H}^{n-1} \leq C \int_{M \setminus F} f(x)|T_F(x)| d\mathcal{H}^{n-1}.$$

We note that $T_F(x) = 0$ for $x \in M \cap \partial F$. It is therefore clear that the above inequality and the condition (ii) on f imply

$$(5.20) \quad \frac{\partial}{\partial s}|E \setminus F_{t,s}| \leq C\varepsilon|F \Delta E|.$$

To prove (5.16) we write $\varphi(t, s) = |F_{t,s} \setminus E| - |E \setminus F_{t,s}|$. From (5.14) it follows that $|F_{t,s} \setminus E| = |F_{t+s} \setminus E|$. From (5.17), (5.18) and (5.20) we conclude that

$$\frac{\partial}{\partial s}\varphi(t, s) = \frac{\partial}{\partial s}(|F_{t+s} \setminus E| - |E \setminus F_{t,s}|) \geq (c - C\varepsilon)|F \Delta E|.$$

Therefore (5.16) follows by choosing ε to be small enough.

Notice that $s(0) = s(1) = 0$. This follows simply from the fact $|F| = |E|$ and (5.16). To be sure that for every t the value $s(t)$ indeed exists we claim that for $t \in (0, 1)$ we have

$$(5.21) \quad -t < s(t) < 1 - t.$$

To that aim we again write $\varphi(t, s) = |F_{t+s} \setminus E| - |E \setminus F_{t,s}|$. Since $F_0 = E$ and $F_1 = F$ we conclude that

$$|F_{t,-t} \setminus E| = 0 \quad \text{and} \quad |F_{t,1-t} \setminus E| = |F \setminus E|.$$

Similarly since $f(x) < 1$ on $M \setminus F$ it holds that

$$|F_{t,-t} \cap E| > 0 \quad \text{and} \quad |F_{t,1-t} \cap E| < |F \setminus E|.$$

Hence $\varphi(t, -t) < 0$ and $\varphi(t, 1 - t) > 0$, and (5.21) follows. Since the flow Φ_Y is uniformly $W^{2,p}$ regular, the function φ is C^2 regular. Therefore the condition (5.16) and implicit function theory imply that (5.15) defines a function $s(\cdot)$ which is C^2 regular.

In order to conclude that $t + s(t)$ defines a parametrization of the interval $[0, 1]$ we show that for every $t \in [0, 1]$

$$(5.22) \quad s'(t) > -1 + \mu,$$

for some $\mu > 0$. Similarly as we obtained (5.19) we use the regularity of \tilde{X}_t to deduce that for every $t \in (0, 1)$ and $s \in (-t, 1 - t)$ we have

$$\frac{\partial}{\partial t} |E \setminus F_{t,s}| = \int_{\partial F_{t,s} \cap E} |\langle Y, \nu_{F_{t,s}} \rangle| d\mathcal{H}^{n-1} \geq c \int_{M \setminus F} |Y(x)| d\mathcal{H}^{n-1} \geq c |F \Delta E|.$$

The above estimate, (5.17) and (5.20) yield

$$\begin{aligned} 0 &= \frac{d}{dt} \varphi(t, s(t)) = \frac{d}{dt} (|F_{t+s(t)} \setminus E| - |E \setminus F_{t,s(t)}|) \\ &= \frac{d}{dt} |F_t \setminus E| \Big|_{t=t+s(t)} (1 + s'(t)) - \frac{\partial}{\partial t} |E \setminus F_{t,s}| \Big|_{s=s(t)} - \frac{\partial}{\partial s} |E \setminus F_{t,s}| \Big|_{s=s(t)} s'(t) \\ &\leq (C \max\{1 + s'(t), 0\} - c + C \sup |s'(t)| \varepsilon) |F \Delta E|. \end{aligned}$$

Consequently we obtain (5.22) when $\varepsilon > 0$ is small enough.

We finally define the flow $\Phi : \Omega \times [0, 1] \rightarrow \Omega$ as

$$(5.23) \quad \Phi(x, t) = \tilde{\Phi}_t(x, s(t)) = \Phi_Y(x, t + s(t)f(x)) \quad \text{for } x \in M,$$

and extend it smoothly to Ω . The sets E_t are defined as

$$(5.24) \quad E_t := \Phi(E, t) = F_{t,s(t)}.$$

By construction E_t satisfies the volume constraint at any time $t \in [0, 1]$. It also satisfies the orthogonality condition, which can be checked similarly as it was done in (5.9) by using the Neumann boundary condition $\langle \nabla f, \nu_\Omega \rangle = 0$ on $M \cap \partial\Omega$.

The map $\Phi_{M \times [0, 1]} \rightarrow F \Delta E$ is C^2 diffeomorphism and we may define projections $\pi : F \Delta E \rightarrow M$ and $T : F \Delta E \rightarrow [0, 1]$ such that for every $y \in F \Delta E$

$$\Phi(\pi(y), T(y)) = y.$$

It is clear that the projections are C^2 regular with uniformly bounded $W^{2,p}$ -norms. The vector field X can be written on $F \Delta E$ as

$$(5.25) \quad X(y) = (1 + s'(T(y)) f(\pi(y))) Y(y).$$

It is then clear that X is C^2 regular with uniformly bounded $W^{2,p}$ -norm. \square

In the next lemma we study the regularity properties of the flow Φ and the vector field X constructed in the previous lemma. The reader is encouraged to skip it in the first reading.

Lemma 5.4. *Suppose that E , F , Φ and X are as in Lemma 5.3 and denote $M_t := \overline{\partial E_t \cap \Omega}$. The vector field $X \in C^2(\Omega; \mathbb{R}^n)$ satisfies*

$$(5.26) \quad |F\Delta E| \leq C \int_M |\langle X, \nu_M \rangle| d\mathcal{H}^{n-1}$$

and

$$(5.27) \quad \|X_{\tau_t}\|_{H^1(M_t)} \leq C \|\langle X, \nu_t \rangle\|_{H^1(M_t)}, \quad \|\operatorname{div}(X)\|_{L^2(M_t)} \leq C \|\langle X, \nu_t \rangle\|_{L^2(M_t)}$$

for some constant C .

Proof. We begin by making some remarks on the regularity of the flow Φ and the vector field X . Notice first that by (5.25) the vector field X can be written as

$$X(y) = \phi(y)Y(y)$$

where ϕ is a C^2 regular function with uniformly bounded $W^{2,p}$ norm and $C^{-1} \leq \phi \leq C$. The estimate $|Y(y)| \leq C|T_F(y)|$ proved in (5.10) yields

$$C^{-1}|T_F(y)| \leq |X(y)| \leq C|T_F(y)| \quad \text{for all } y \in F\Delta E.$$

We recall that for $x \in M$, $T_F(x)$ denotes the time needed from x to M_F along the primitive vector field Z and $\|T_F\|_{W^{2,p}} \leq C\|F, E\|_{W^{2,p}}$. Since T_F is constant along the trajectories it follows from the above estimate that for every $t \in [0, 1]$

$$(5.28) \quad C^{-1}|X(x)| \leq |X(\Phi(x, t))| \leq C|X(x)| \quad \text{for all } x \in M.$$

Denote $\Phi_t(\cdot) = \Phi(\cdot, t)$ and by ν_{M_t} the normal vector of M_t . For every $t \in [0, 1]$ the map $\Phi(\cdot, t)$ is C^2 diffeomorphism. We may show, exactly as we did (5.13) in Lemma 5.3, that

$$(5.29) \quad \|\Phi(\cdot, t) - Id\|_{W^{2,p}(M)} \leq C\|X\|_{W^{2,p}} \leq C\|T_F\|_{W^{2,p}}$$

and

$$(5.30) \quad |\langle X, \nu_{M_t} \rangle| \geq (1 - 3\varepsilon)|X|.$$

We proceed by showing (5.26). We use the same argument as we did with (5.18) and (5.19) and calculate

$$\frac{d}{dt}|E_t\Delta E| = \frac{d}{dt}(|E_t \setminus E| + |E \setminus E_t|) = \int_{M_t} |\langle X, \nu_{M_t} \rangle| d\mathcal{H}^{n-1} \leq C \int_M |\langle X, \nu_M \rangle| d\mathcal{H}^{n-1}$$

where we have used the regularity of the flow, i.e., $|J_M\Phi_t| \leq C$, (5.28) and (5.30). The previous calculation then yields

$$|F\Delta E| = \int_0^1 \frac{d}{dt}|E_t\Delta E| dt \leq C \int_M |\langle X, \nu_M \rangle| d\mathcal{H}^{n-1}.$$

The first estimate in (5.27) follows from (5.29) and (5.30) and from the $W^{2,p}$ -regularity of the sets E_t . In fact the proof is exactly the same as the one of Lemma 7.1 in [1] and it will therefore be omitted here.

To prove the second inequality in (5.27) we estimate the divergence by using (5.12)

$$|\operatorname{div}(X)| = |\operatorname{div}(\phi Y)| \leq |\langle \nabla \phi, Y \rangle| + |\phi \operatorname{div}(Y)| \leq |\nabla \phi||Y| + C|\phi Y| \leq C|X|.$$

Hence, the second inequality in (5.27) follows from (5.30). \square

In the following lemma we show by continuity that in a neighbourhood of a critical point with positive second variation the quadratic form remains positive (2.6). Notice that we have to compute the quadratic form for sets which are not critical.

Lemma 5.5. *Suppose that E is as in Theorem 2.3 and $p > n$. There is $\delta > 0$ such that for any C^2 -regular set F with $\|E, F\|_{W^{2,p}} \leq \delta$ and every $\varphi \in H^1(M_F)$ with $\int_{M_F} \varphi d\mathcal{H}^{n-1} = 0$ it holds*

$$\partial^2 J(F)[\varphi] \geq \frac{c_0}{2} \|\varphi\|_{H^1(\partial F)}^2.$$

Here $M_F = \overline{\partial F \cap \Omega}$, $\partial^2 J(F)[\varphi]$ is defined in (2.6) and c_0 is the constant from Lemma 5.1.

Proof. We argue by contradiction and assume that there are F_k with $\|E, F_k\|_{W^{2,p}} \rightarrow 0$ and $\varphi_k \in H^1(M_k)$ with $\int_{M_k} \varphi_k d\mathcal{H}^{n-1} = 0$, which by scaling we may assume to satisfy $\|\varphi_k\|_{H^1(M_k)} = 1$, such that

$$(5.31) \quad \lim_{k \rightarrow \infty} \partial^2 J(F_k)[\varphi_k] \leq \frac{c_0}{2}$$

where $M_k = \overline{\partial F_k \cap \Omega}$. We recall the formula for $\partial^2 J(F_k)$

$$(5.32) \quad \begin{aligned} \partial^2 J(F_k)[\varphi_k] &= \int_{M_k} (|D_{\tau_k} \varphi_k|^2 - |B_{M_k}|^2 \varphi_k^2) d\mathcal{H}^{n-1} - \int_{M_k \cap \partial \Omega} B_{\partial \Omega}(\nu_k, \nu_k) \varphi_k^2 d\mathcal{H}^{n-2} \\ &+ 8\gamma \int_{M_k} \int_{M_k} G(x, y) \varphi_k(x) \varphi_k(y) d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \\ &+ 4\gamma \int_{M_k} \langle \nabla v_{F_k}, \nu_k \rangle \varphi_k^2 d\mathcal{H}^{n-1}, \end{aligned}$$

where ν_k is the unit normal of F_k . We will show that there is $\varphi \in H^1(M)$ with $\|\varphi\|_{H^1(M)} = 1$ and $\int_M \varphi d\mathcal{H}^{n-1} = 0$ such that

$$\lim_{k \rightarrow \infty} \partial^2 J(F_k)[\varphi_k] \geq \partial^2 J(E)[\varphi].$$

up to a subsequence. This and (5.31) then contradicts Lemma 5.1.

Since $\|E, F_k\|_{W^{2,p}} \rightarrow 0$ there are C^2 -diffeomorphisms $\Phi_k : E \rightarrow F_k$ such that $\|\Phi_k - Id\|_{W^{2,p}} \rightarrow 0$. By compactness there exists $\varphi \in H^1(\partial E)$ with $\|\varphi\|_{H^1(\partial E)} = 1$ and $\int_{\partial E} \varphi d\mathcal{H}^{n-1} = 0$ such that, up to a subsequence,

$$\varphi_k \circ \Phi_k \rightharpoonup \varphi \quad \text{weakly in } H^1(M),$$

where $M = \overline{\partial E \cap \Omega}$. In particular $\varphi_k \circ \Phi_k \rightarrow \varphi$ strongly in $L^2(M)$. We also conclude that $\nu_k \circ \Phi_k \rightarrow \nu$ uniformly on M where ν is the unit normal of E . Therefore by the weak lower semicontinuity we obtain the convergence of the first term in (5.32)

$$\lim_{k \rightarrow \infty} \int_{M_k} |D_{\tau_k} \varphi_k|^2 d\mathcal{H}^{n-1} \geq \int_M |D_{\tau} \varphi|^2 d\mathcal{H}^{n-1}.$$

We proceed by observing the following convergences

$$B_{M_k} \circ \Phi_k \rightarrow B_M \quad \text{in } L^p(M) \quad \text{and} \quad v_{F_k} \rightarrow v_E \quad \text{in } C^{1,\alpha}(\Omega).$$

Indeed, the first one follows trivially from the $W^{2,p}$ -convergence of F_k and the second one follows from the uniform $C^{1,\alpha}$ -regularity given by the equation (1.4). Therefore we obtain the convergence of the second and the last term in (5.32)

$$\int_{M_k} |B_{M_k}|^2 \varphi_k^2 d\mathcal{H}^{n-1} \rightarrow \int_M |B_M|^2 \varphi^2 d\mathcal{H}^{n-1}$$

and

$$\int_{M_k} \langle \nabla v_{F_k}, \nu_k \rangle \varphi_k^2 d\mathcal{H}^{n-1} \rightarrow \int_M \langle \nabla v_E, \nu \rangle \varphi^2 d\mathcal{H}^{n-1}.$$

By compactness of the trace operator we have that $\varphi_k \circ \Phi_k \rightarrow \varphi$ on $L^2(M \cap \partial\Omega)$. Hence we deduce the convergence of the boundary terms

$$\int_{M_k \cap \partial\Omega} B_{\partial\Omega}(\nu_k, \nu_k) \varphi_k^2 d\mathcal{H}^{n-2} \rightarrow \int_{M \cap \partial\Omega} B_{\partial\Omega}(\nu, \nu) \varphi^2 d\mathcal{H}^{n-2}.$$

We have yet to show

$$(5.33) \quad \int_{M_k} \int_{M_k} G(x, y) \varphi_k(x) \varphi_k(y) d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \rightarrow \int_M \int_M G(x, y) \varphi(x) \varphi(y) d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y)$$

to conclude the proof. Denote by $J_M \Phi_k$ the Jacobian of Φ_k on M and notice that $J_M \Phi_k \rightarrow 1$ uniformly on M . For simplicity we denote $\tilde{\varphi}_k = \varphi_k \circ \Phi_k$. Firstly we deduce

$$\begin{aligned} & \left| \int_{M_k} \int_{M_k} G(x, y) \varphi_k(x) \varphi_k(y) d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) - \int_M \int_M G(x, \Phi_k(y)) \varphi(x) \tilde{\varphi}_k(y) J_M \Phi_k(y) d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \right| \\ & \leq \int_M |J_M \Phi_k(y) \tilde{\varphi}_k(y)| \int_M |G(\Phi_k(x), \Phi_k(y)) \tilde{\varphi}_k(x) J_M \Phi_k(x) - G(x, \Phi_k(y)) \varphi(x)| d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \\ & \leq C \left(\int_M \left(\int_M |G(\Phi_k(x), \Phi_k(y)) \tilde{\varphi}_k(x) J_M \Phi_k(x) - G(x, \Phi_k(y)) \varphi(x)| d\mathcal{H}^{n-1}(x) \right)^2 d\mathcal{H}^{n-1}(y) \right)^{1/2} \\ & \leq C \left(\int_M \int_M |G(\Phi_k(x), \Phi_k(y)) J_M \Phi_k(x) - G(x, \Phi_k(y))|^2 d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \right)^{1/2} + C \left(\int_M |\tilde{\varphi}_k - \varphi|^2 d\mathcal{H}^{n-1} \right)^{1/2} \\ & \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Similarly we obtain

$$\int_M \int_M G(x, \Phi_k(y)) \varphi(x) \tilde{\varphi}_k(y) J_M \Phi_k(y) d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \rightarrow \int_M \int_M G(x, y) \varphi(x) \varphi(y) d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y).$$

These two estimates yield (5.33) and the lemma is proved. \square

We are now ready to prove Proposition 5.2.

Proof of the Proposition 5.2. Without loss of generality we may assume that F is a C^2 regular and $M_F = \overline{\partial F \cap \Omega}$ is a C^2 -manifold with boundary and it meets $\partial\Omega$ orthogonally. Moreover we assume that $\|F, E\|_{W^{2,p}} \leq \delta$ where $\delta > 0$ is chosen later.

Suppose that X is the vector field and Φ is the associated flow given by Lemma 5.3. For every $t \in [0, 1]$ we denote $E_t = \Phi(E, t)$, $M_t = \overline{\partial E_t \cap \Omega}$ and ν_t as the unit normal of E_t . Since E is a critical set we have

$$\frac{d}{dt} J(E_t) \Big|_{t=0} = 0.$$

Moreover Lemma 5.1 yields $\frac{d^2}{dt^2} J(E_t) \Big|_{t=0} \geq \frac{c_0}{2} \|\varphi\|_{H^1(M)}^2$ where $\varphi = \langle X, \nu \rangle$ for some constant $c_0 > 0$. The idea of the proof is to show by a continuity argument that when $\delta > 0$ is chosen small enough we have

$$(5.34) \quad \frac{d^2}{dt^2} J(E_t) \geq \frac{c_0}{4} \|\varphi_t\|_{H^1(M_t)}^2,$$

for every $t \in [0, 1]$ where $\varphi_t = \langle X, \nu_t \rangle$.

The result follows from the estimate (5.34). Indeed when $\delta > 0$ is small we have by (5.28) and (5.30) that

$$\int_{M_t} |\varphi_t|^2 d\mathcal{H}^{n-1} = \int_M J_M \Phi_t |\langle X, (\nu_t \circ \Phi_t) \rangle|^2 d\mathcal{H}^{n-1} \geq c \int_M |\langle X, \nu \rangle|^2 d\mathcal{H}^{n-1}.$$

Therefore the estimates (5.34) and (5.26) from Lemma 5.4 yield

$$J(F) - J(E) = \int_0^1 (1-t) \frac{d^2}{dt^2} J(E_t) dt \geq c \|\varphi\|_{L^2(M)}^2 \geq c |F \Delta E|^2,$$

and the claim follows. Hence, we need to proof (5.34).

The formula for $\frac{d^2}{dt^2} J(E_t)$ in Proposition 4.1 can be written as

$$\begin{aligned} \frac{d^2 J(E_t)}{dt^2} &= \partial^2 J(E_t)[\varphi_t] - \int_{M_t} (H_{\partial E_t} + 4\gamma v_{E_t}) \operatorname{div}_{\tau_t} (X_{\tau_t} \langle X, \nu_t \rangle) d\mathcal{H}^{n-1} \\ &\quad + \int_{M_t} (H_{\partial E_t} + 4\gamma v_{E_t}) \operatorname{div}(X) \langle X, \nu_t \rangle d\mathcal{H}^{n-1}, \end{aligned}$$

where $\partial^2 J(E_t)[\cdot]$ is the quadratic form defined in (2.6) and $\varphi_t = \langle X, \nu_t \rangle$ as before. Notice that since the flow preserves the volume we have

$$\int_{M_t} \varphi_t d\mathcal{H}^{n-1} = \int_{M_t} \langle X, \nu_t \rangle d\mathcal{H}^{n-1} = 0.$$

When $\delta > 0$ is small enough Lemma 5.5 yields

$$(5.35) \quad \partial^2 J(E_t)[\varphi_t] \geq \frac{c_0}{2} \|\varphi_t\|_{H^1(M_t)}^2.$$

We proceed by showing that when $\delta > 0$ is small enough we have

$$(5.36) \quad S_t := \left| \int_{M_t} (H_{\partial E_t} + 4\gamma v_{E_t}) \operatorname{div}_{\tau_t} (X_{\tau_t} \langle X, \nu_t \rangle) d\mathcal{H}^{n-1} \right| \leq \frac{c_0}{8} \|\varphi_t\|_{H^1(M_t)}^2.$$

Indeed, since E solves (2.4) we deduce $(H_{\partial E_t} + 4\gamma v_{E_t}) \circ \Phi(\cdot, t) \rightarrow \lambda$ in $L^p(M)$. Therefore when $\delta > 0$ is small we have for small $\varepsilon > 0$

$$\begin{aligned} S_t &= \left| \int_{M_t} (H_{\partial F_t} + 4\gamma v_{F_t} - \lambda) \operatorname{div}_{\tau_t} (X_{\tau_t} \langle X, \nu_t \rangle) d\mathcal{H}^{n-1} \right| \\ &\leq \left(\int_{M_t} (H_{\partial F_t} + 4\gamma v_{F_t} - \lambda)^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left(\int_{M_t} (\operatorname{div}_{\tau_t} (X_{\tau_t} \langle X, \nu_t \rangle))^{\frac{p}{p-1}} d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p}} \\ &\leq \varepsilon \left(\|D_{\tau_t} X_{\tau_t}\|_{L^2(M_t)} \|\langle X, \nu_t \rangle\|_{L^{\frac{2p}{p-2}}(M_t)} + \|D_{\tau_t} \langle X, \nu_t \rangle\|_{L^2(M_t)} \|X_{\tau_t}\|_{L^{\frac{2p}{p-2}}(M_t)} \right) \\ &\leq C\varepsilon \|D_{\tau_t} X_{\tau_t}\|_{H^1(M_t)} \|\langle X, \nu_t \rangle\|_{H^1(M_t)} \end{aligned}$$

where the last inequality follows from Sobolev inequality and the fact that $p > n$. The estimate (5.36) then follows from (5.27).

Similarly we prove that when $\delta > 0$ is small we obtain

$$(5.37) \quad R_t := \left| \int_{M_t} (H_{\partial F_t} + 4\gamma v_{F_t}) \operatorname{div}(X) \langle X, \nu_t \rangle d\mathcal{H}^{n-1} \right| \leq \frac{c_0}{8} \|\varphi_t\|_{H^1(M_t)}^2.$$

The volume constraint $|F_t| = |E|$ yields

$$\int_{M_t} \operatorname{div}(X) \langle X, \nu_t \rangle d\mathcal{H}^{n-1} = 0$$

as it was discussed earlier after Proposition 4.1. As before using the fact that E solves (2.4) we obtain by Hölder and Sobolev inequalities

$$\begin{aligned} R_t &= \left| \int_{M_t} (H_{\partial F_t} + 4\gamma v_{F_t} - \lambda) \operatorname{div}(X) \langle X, \nu_t \rangle d\mathcal{H}^{n-1} \right| \\ &\leq \left(\int_{M_t} (H_{\partial F_t} + 4\gamma v_{F_t} - \lambda)^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left(\int_{M_t} (\operatorname{div}(X) \langle X, \nu_t \rangle)^{\frac{p}{p-1}} d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p}} \\ &\leq C\varepsilon \|\operatorname{div}(X)\|_{L^2(M_t)} \|\langle X, \nu_t \rangle\|_{H^1(M_t)}, \end{aligned}$$

for any $\varepsilon > 0$ when $\delta > 0$ is small enough. The estimate (5.37) follows from (5.27).

The key estimate (5.34) now follows from (5.35), (5.36) and (5.37) and the proposition is proved. \square

6. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 2.3. As it was mentioned in the introduction we will effectively use the regularity of ω -minimizers to rule out those competing sets which are not regular. This idea goes back to the work by Cicalese and Leonardi [8] and by Fusco and Morini [9].

We begin with a simple lemma.

Lemma 6.1. *Suppose that $E \subset \Omega$ is C^2 -regular and satisfies the orthogonality condition (2.5). There is a constant C depending only on E such that for every $F \in BV(\Omega)$ we have*

$$J(E) \leq J(F) + C|F\Delta E|.$$

Proof. By the Lipschitz continuity of the non-local part (3.2) we have

$$J(E) \leq J(F) + P_\Omega(E) - P_\Omega(F) + C|F\Delta E|.$$

On the other hand by the assumptions on E we may construct a C^1 vector field X on Ω such that $X = \nu_E$ on $M = \overline{\partial E \cap \Omega}$, $\langle X, \nu_\Omega \rangle = 0$ on $\partial\Omega$ and $\|X\|_{L^\infty} \leq 1$. Therefore

$$\begin{aligned} P_\Omega(E) - P_\Omega(F) &\leq \int_{\partial(E \cap \Omega)} \langle X, \nu \rangle d\mathcal{H}^{n-1}(x) - \int_{\partial(F \cap \Omega)} \langle X, \nu \rangle d\mathcal{H}^{n-1}(x) \\ &= \int_{E \cap \Omega} \operatorname{div} X \, dx - \int_{F \cap \Omega} \operatorname{div} X \, dx \\ &\leq \|\operatorname{div} X\|_{L^\infty} |F\Delta E|. \end{aligned}$$

\square

We remark that the above proof also yields that E is an ω -minimizer in the sense of Definition 3.4.

The next lemma states that the convergence of mean curvatures implies $W^{2,p}$ convergence for any $p > 1$. It is very similar to Lemma 7.2 in [1].

Lemma 6.2. *Let $p > 1$. Suppose that E is as in Theorem 2.3 and there are diffeomorphisms $\Phi_k : E \rightarrow F_k$ such that $\|\Phi_k - Id\|_{C^{1,\alpha}} \rightarrow 0$ and*

$$H_{\partial F_k}(\Phi_k(\cdot)) \rightarrow H_{\partial E}(\cdot) \quad \text{in } L^p.$$

Then there are diffeomorphisms $\tilde{\Phi}_k : E \rightarrow F_k$ such that $\|\tilde{\Phi}_k - Id\|_{W^{2,p}} \rightarrow 0$.

Sketch of the proof. Since F_k are $C^{1,\alpha}$ manifolds with boundary they can be presented locally as a graph of $C^{1,\alpha}$ functions. Away from the relative boundary $\overline{\partial F_k \cap \Omega}$ the convergence follows as in ([1], Lemma 7.2). On the boundary we follow a standard argument and flatten the boundary $\partial\Omega$ locally to half space and then use similar elliptic regularity estimate as in the interior case. \square

We are now ready to prove the main result of the paper. As in [1, 5] the proof is divided into two steps where we first prove that E is a local minimizer with respect to Hausdorff distance and then sharpen the result to obtain the local minimality with respect to L^1 distance with the quantitative estimate.

Proof of Theorem 2.3. Step 1: First we prove that E is a local minimizer in L^∞ -topology, i.e., there is $\delta > 0$ such that for every $F \in BV(\Omega)$ with $|F| = |E|$ and $F \subset \{x \in \Omega \mid \text{dist}(x, E) < \delta\}$ it holds

$$J(F) \geq J(E).$$

For simplicity we denote $\mathcal{I}_\delta(E) := \{x \in \Omega \mid \text{dist}(x, E) < \delta\}$

We argue by contradiction and assume there are $E_k \subset \Omega$ such that $|E_k| = |E|$, $E_k \subset \mathcal{I}_{1/k}(E)$ and

$$J(E_k) < J(E).$$

We will use the signed distance function $d_{\Omega, E}(\cdot)$ of E on Ω constructed in the first step of Lemma 5.3. Denote the sets $\{x \in \Omega \mid d_{\Omega, E}(x) < 1/k\} \cup E$ by U_k . The reader needs only to know that these sets are C^3 regular and satisfy

- (i) $\mathcal{I}_{1/k'}(E) \subset U_k$, for some k'
- (ii) $\|U_k, E\|_{C^2} \rightarrow 0$ in a sense of definition (5.1) and
- (iii) U_k satisfies the orthogonality condition (2.5).

As in [1] we replace the contradicting sequence (E_k) by (F_k) which minimizes

$$(6.1) \quad J(F) \quad \text{with constraint } |F| = |E| \text{ and } F \subset U_k.$$

By the contradiction assumption $J(F_k) < J(E)$ for every k and obviously $F_k \rightarrow E$ in Hausdorff topology.

Let us show that F_k are ω -minimizers with uniform constants Λ and r . To that aim let $G \subset \Omega$ be a regular set such that $G \Delta F_k \subset B_r(x_0)$. Let us divide G into two parts

$$G \cap U_k \quad \text{and} \quad G \cup U_k.$$

Since F_k minimizes (1.3) in U_k with a volume constraint we obtain as in the proof of Proposition 3.4 that

$$(6.2) \quad P_{B_r(x_0)}(F_k) \leq P_{B_r(x_0)}(G \cap U_k) + C|(G \Delta F_k) \cap U_k|$$

for some C . Moreover, since U_k are uniformly C^3 regular and satisfy the orthogonality condition they are ω -minimizers, as we remarked after the proof of Lemma 6.1, and therefore

$$(6.3) \quad P_{B_r(x_0)}(U_k) \leq P_{B_r(x_0)}(G \cup U_k) + C|G \setminus U_k|,$$

for some $C > 0$. Since

$$P_{B_r(x_0)}(G \cap U_k) + P_{B_r(x_0)}(G \cup U_k) \leq P_{B_r(x_0)}(G) + P_{B_r(x_0)}(U_k)$$

the estimates (6.2) and (6.3) yield

$$P_{B_r(x_0)}(F_k) \leq P_{B_r(x_0)}(G) + \Lambda|F_k \Delta G|$$

for some large Λ and the ω -minimality is proved. By the ω -minimizing property of F_k and by Theorem 3.5 we conclude that $F_k \rightarrow E$ in $C^{1,\alpha}$ and that F_k satisfy the orthogonality condition (2.5).

Denote the relative boundaries as usual by $M = \overline{\partial E} \cap \overline{\Omega}$ and $M_k = \overline{\partial F_k} \cap \overline{\Omega}$. The Euler-Lagrange equation for F_k reads as

$$(6.4) \quad \begin{cases} H_{\partial F_k} + 4\gamma v_{F_k} = \lambda_k & \text{on } M_k \cap U_k, \\ H_{\partial F_k} + 4\gamma v_E + \gamma_k = \lambda & \text{on } M_k \setminus U_k, \end{cases}$$

where γ_k are some remainder terms with $\gamma_k \rightarrow 0$ as $k \rightarrow \infty$. Since F_k are C^1 regular Proposition 3.2 yields $\|H_{\partial F_k}\|_{L^\infty} \leq \Lambda$. Moreover the equation (1.4) implies

$$(6.5) \quad v_{F_k} \rightarrow v_E \text{ in } C^1(\Omega).$$

Next we show that

$$(6.6) \quad H_{\partial F_k}(\Phi_k(\cdot)) \rightarrow H_{\partial E}(\cdot) \quad \text{on } L^p(M),$$

for any $p > 1$. Indeed, consider a vector field $X \in C^1(\Omega, \mathbb{R}^n)$ as in the proof of Lemma 6.1, i.e., $X = \nu_M$ on M and $\langle X, \nu_\Omega \rangle = 0$ on $\partial\Omega$. We multiply the equation (6.4) by $\langle X, \nu_{M_k} \rangle$, integrate over M_k and then integrate by parts and use the $C^{1,\alpha}$ -convergence to deduce

$$\begin{aligned} & \int_{M_k \cap U_k} (\lambda_k - 4\gamma v_{F_k}) \langle X, \nu_{M_k} \rangle d\mathcal{H}^{n-1} + \int_{M_k \setminus U_k} (\lambda - 4\gamma v_E - \gamma_k) \langle X, \nu_M \rangle d\mathcal{H}^{n-1} \\ &= \int_{M_k \cap \Omega} H_{\partial F_k} \langle X, \nu_{F_k} \rangle d\mathcal{H}^{n-1} = \int_{M_k \cap \Omega} \operatorname{div}_{M_k} X d\mathcal{H}^{n-1} \\ &\rightarrow \int_{M \cap \Omega} \operatorname{div}_M X d\mathcal{H}^{n-1} \\ &= \int_{M \cap \Omega} (\lambda - 4\gamma v_E) d\mathcal{H}^{n-1}, \end{aligned}$$

where the last equality follows from the Euler-Lagrange equation (2.4) for E and from $X = \nu_E$ on $\partial E \cap \Omega$. The $C^{1,\alpha}$ -convergence of F_k and (6.5) then imply that either $\lambda_k \rightarrow \lambda$, or $\mathcal{H}^{n-1}(M_k \setminus U_k) \rightarrow 0$. In either case we obtain (6.6) since $H_{\partial F_k}$ are uniformly bounded in L^∞ .

From (6.6) and Lemma 6.2 we deduce $\|F_k, E\|_{W^{2,p}} \rightarrow 0$. Since F_k satisfy the orthogonality condition we may use Proposition 5.2 to conclude

$$J(F_k) \geq J(E)$$

when k is large. This contradicts with the minimality of F_k (6.1) since by the contradiction assumption we have $J(F_k) < J(E)$.

Step 2: We are ready to prove Theorem 2.3.

As before we argue by contradiction and assume that there are $E_k \subset \Omega$ such that $|E_k| = |E|$, $|E_k \Delta E| \rightarrow 0$ and

$$J(E_k) < J(E) + \frac{c_1}{4} |E_k \Delta E|^2,$$

for $c_1 > 0$ from Proposition 5.2. Denote $\varepsilon_k := |E_k \Delta E|$. As in [1] we replace the contradicting sequence (E_k) by (F_k) which minimizes

$$(6.7) \quad \min\{J(F) + C\sqrt{(|F \Delta E| - \varepsilon_k)^2 + \varepsilon_k} : F \subset \Omega \text{ with } |F| = |E|\},$$

for some constant C which will be chosen later.

By compactness we may assume that, up to a subsequence, $F_k \rightarrow F_0$ in L^1 and that F_0 solves

$$\min\{J(F) + C|F \Delta E| : F \subset \Omega \text{ with } |F| = |E|\}.$$

By choosing C large enough it follows from Lemma 6.1 that $F_0 = E$. In particular $F_k \rightarrow E$ in L^1 .

As in step 1 we observe that every F_k is an ω -minimizer with uniform constants Λ and r_0 . In fact, since there are no obstacle in (6.1) this observation follows exactly as in the proof of the Proposition 3.4. Therefore Theorem 3.5 implies that $F_k \rightarrow E$ in $C^{1,\alpha}$ and that F_k are $C^{1,\alpha}$ -manifolds with boundary for k large and satisfies the orthogonality condition (2.5).

The Euler-Lagrange equation for F_k reads as

$$(6.8) \quad \begin{cases} H_{\partial F_k} + 4\gamma v_{F_k} + C \frac{(|F \Delta E| - \varepsilon_k)}{\sqrt{(|F \Delta E| - \varepsilon_k)^2 + \varepsilon_k}} = \lambda_k & \text{on } M_k \setminus M, \\ H_{\partial F_k} + 4\gamma v_E = \lambda & \text{on } M_k \cap M. \end{cases}$$

Since the mean curvature $H_{\partial F_k}$ is bounded we conclude that M_k is $W^{2,p}$ -regular.

Using the minimality of F_k and the contradiction assumption and step 1 we obtain

$$\begin{aligned} J(F_k) + C\sqrt{(|F_k \Delta E| - \varepsilon_k)^2 + \varepsilon_k} &\leq J(E_k) + C\sqrt{\varepsilon_k} \\ &\leq J(E) + \frac{c_0}{8}\varepsilon_k^2 + C\sqrt{\varepsilon_k} \\ &\leq J(F_k) + \frac{c_0}{8}\varepsilon_k^2 + C\sqrt{\varepsilon_k}, \end{aligned}$$

where the last inequality follows from the L^∞ -local minimality proved in step 1. The previous inequality yields

$$(6.9) \quad \lim_{k \rightarrow \infty} \frac{(|F_k \Delta E| - \varepsilon_k)}{\varepsilon_k} = 0.$$

In particular

$$\lim_{k \rightarrow \infty} \frac{(|F \Delta E| - \varepsilon_k)}{\sqrt{(|F \Delta E| - \varepsilon_k)^2 + \varepsilon_k}} = 0.$$

Arguing exactly as in step 1 we conclude from the equation (6.8) and from (6.9) that $\|F_k, E\|_{W^{2,p}} \rightarrow 0$ for any $p > 1$. Hence we may use Proposition 5.2 to conclude

$$J(F_k) \geq J(E) + c_1 |F_k \Delta E|^2.$$

However the minimality of F_k , the contradiction assumption and (6.9) yield

$$J(F_k) \leq J(E_k) \leq J(E) + \frac{c_1}{4}\varepsilon_k^2 \leq J(E) + \frac{c_1}{2} |F_k \Delta E|^2$$

which is a contradiction. \square

Proof of the Corollary 2.4. We obtain Corollary 2.4 immediately from Theorem 2.3 arguing as in Theorem 6.3 in [1]. \square

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF JYVÄSKYLÄ, FINLAND
E-mail address: vesa.julin@jyu.fi

DIPARTIMENTO DI MATEMATICA, SECONDA UNIVERSITA DI NAPOLI, CASERTA, ITALY
E-mail address: giovanni.pisante@unina2.it